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CONTENTS

	PAGE
On a theorem of Torelli. By ALDO ANDREOTTI,	801
The spectra of multiplier transforms on l^p. By A. DEVINATZ and I. I. HIRSCHMAN, JR.,	829
The integral representations of quadratic forms over local fields. By O. T. O'MEARA,	843
On power multiplicative norms. By SILVIO AURORA,	879
The suspension of a loop space. By W. D. BARCUS and J.-P. MEYER,	895
An interpolation problem for bounded analytic functions. By LENNART CARLESON,	921
Continuity of solutions of parabolic and elliptic equations. By J. NASH,	931
The algebraic simplicity of certain groups of homeomorphisms. By R. D. ANDERSON,	955
Applications of the theory of Morse to symmetric spaces. By RAOUL BOTT and HANS SAMELSON,	964

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ON A THEOREM OF TORELLI.*

By ALDO ANDREOTTI.

Denote by X an algebraic non-singular curve of genus g in some (complex) projective space $P(\boldsymbol{C})$. Choose on X a base $\gamma_1, \cdots, \gamma_{2g}$ for $H_1(X, Z)$ such that $(I(\gamma_i, \gamma_j)) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and a base $\omega_1, \cdots, \omega_g$ for $H^0(X, \Omega^1)$ (differentials of the first kind). Denote by $\omega(X)$ the matrix of the periods $\omega = \left(\int_{\gamma_i} \omega_j\right)$.

The classical form of Torelli's theorem is as follows: *two curves X, X' of the same genus g are birationally equivalent if (and only if) the bases $\{\gamma_i\}$ $\{\omega_j\}$ may be so chosen that $\omega(X) = \omega(X')$.*[1]

In a recent paper, A. Weil [9(c)] has given an abstract form of this statement which is as follows: *two curves X, X' are birationally equivalent if (and only if) there exists an isomorphism of the Jacobian variety $J(X)$ of X onto the Jacobian variety $J(X')$ of X' which carries the canonical polarization of $J(X)$ into the canonical polarization of $J(X')$.*

Now it is possible to give to Torelli's theorem a simpler and (apparently) more general form from which the classical as well as the abstract formulation can be easily deduced.

The statement is as follows: *The necessary and sufficient condition for two curves X, X' of the same genus g to be birationally equivalent is that their symmetric products $(X)^{(g-1)}$, $(X')^{(g-1)}$ be birationally equivalent.*

The previously given forms of Torelli's theorem follow at once from the remark that the symmetric product $(X)^{(g-1)}$ is birationally equivalent to the canonical Θ-divisor on $J(X)$,[2] and the fact, proved by A. Weil,[3] that the canonical polarization determines uniquely the Θ-divisor.

---

* Received February 24, 1958.

[1] Once the base $\{\gamma_i\}$ is chosen, we may normalize the base $\{\omega_j\}$ in such a way that $\omega(X)$ is of the form (I, A). Torelli's theorem may then be expressed by saying that *the necessary and sufficient condition for two curves X, X' to be birationally equivalent is that the matrices $\omega(X) = (I, A)$, $\omega(X') = (I, A')$ be congruent modulo a transformation of Siegel's modular group.*

[2] See A. Weil [9, (b)], p. 71 (Prop. 15).

[3] A. Weil [9, (c)], Theorem 1. In the classical formulation, from $\omega(X) = \omega(X')$

We will give here a proof of the last statement.

In the first two sections we will recall some more or less known properties of the symmetric product of a curve. The proof of the theorem will occupy section 3. The proof for the case of characteristic zero is given first, since for that case it is a straightforward consequence of the previous sections; we will deal with the general case in the remaining paragraphs. The theory of abelian varieties is not used. During the preparation of the last part of this paper, I had some very helpful conversations on the subject with J. P. Serre, to whom I wish to express my gratitude.

Section 1. Symmetric products.[4]

1. Preliminaries. a) Let X be an algebraic abstract variety defined over a field k which will be supposed algebraically closed. The sheaf of local rings on X will be denoted by $\mathcal{O}(X)$.

On the cartesian product $X \times \cdots \times X = (X)^q$ of q copies of X, there exists a natural algebraic structure. On $(X)^q$, the symmetric group G_q operates by permutation of the coordinate points; each operator is a biregular transformation. For any subgroup $\Gamma \subset G_q$, we may consider the quotient space $(X)^{(\Gamma)} = (X)^q/\Gamma$, and provide it with the natural "ring-structure" in which the sheaf of local rings is defined by the presheaf of local sections of $\mathcal{O}(X^q)$ which are invariant under the operations of Γ.

If any set of q points of X is contained in an affine open subset of X, then the space $(X)^{(\Gamma)}$ has the structure of an algebraic variety. If X is projective, $(X)^{(\Gamma)}$ is also projective. If X is locally normal, so is $(X)^{(\Gamma)}$.[5]

If $\Gamma = G_q$, we will denote by $(X)^{(q)}$ the space $(X)^{(\Gamma)}$.

b) We want to study the singularities of the space $(X)^{(\Gamma)}$. We first note the following lemma (although not strictly necessary for our immediate purposes).

LEMMA. *Let Y be an affine variety defined over k and let Γ be a finite group of linear automorphisms of Y. Denote by y a point of Y and by $p(y)$*

$= (I, A)$, it follows that on $J(X) = J(X')$ the Θ-divisors are given by the same equation $\theta(u) = \sum_{m=-\infty}^{+\infty} \exp({}^t mAm + 2\,{}^t mu) = 0$, ${}^t m = (m_1, \ldots, m_g)$, $m_i \in Z$, ${}^t u = (u_1, \ldots, u_g)$, $u_i = \int \omega_i$. The birational equivalence of $(X)^{(g-1)}$ with the Θ-divisor can be deduced, then, from a classical theorem of Riemann.

[4] This section is mainly expository. All the results contained in it are known. They are gathered together for the convenience of the reader.

[5] J. P. Serre [7, (b)], Ch. III, n. 14.

the natural projection of y on Y/Γ. If the point y has as many transforms as there are elements in Γ, then y and $p(y)$ are both simple or singular.

Denote by $I(Y)=k[x_1,\cdots,x_n]$ the coordinate ring of Y and let $j_1,\cdots,j_s$ be a set of generators for the subring of Γ-invariants; the ring $I(Y/\Gamma)$ of coordinates on Y/Γ may be assumed to be the ring $k[j_1,\cdots,j_s]$.

Since the j's are polynomials in the x's, we have relations (α) $dj_h=\sum(\partial j_h/\partial x_i)_y\,dx_i$. Conversely, since k is infinite, we may suppose that the transforms of y have all the i-th coordinates x_i two by two distinct $1\leqq i\leqq n$. The x_i's are integers over $k[j_1,\cdots,j_s]$ and hence satisfy equations of the type $x_i^q+a_1(j)x_i^{q-1}+\cdots+a_q(j)\equiv f_i(x_i,j)=0$. By the above hypothesis, we must have $(\partial f_i/\partial x_i)_y\neq 0$; therefore the relations (α) may be solved with the expressions (β) $dx_i=-(\partial f_i/\partial x_i)_y^{-1}\sum(\partial f_i/\partial j_h)_{p(y)}dj_h$. Now $\dim Y=\dim Y/\Gamma$, and this enables us to conclude that if P describes the ideal $\mathfrak{p}(Y)$ of the polynomials in $k[X_1,\cdots,X_n]$ vanishing on Y, and Q describes the ideal $\mathfrak{p}(Y/\Gamma)$ of the polynomials in $k[J_1,\cdots,J_s]$ vanishing on Y/Γ, there are as many solutions linearly independent for the system $\sum(\partial P/\partial X_i)_y\,dX_i=0$ as for the system $\sum(\partial Q/\partial J_h)_{p(y)}dJ_h=0$. This proves the lemma.

If we apply the lemma to the case in which Y is an open subset of $(X)^q$ invariant under Γ, we easily infer the following:

PROPOSITION 1. *If X is a projective variety of dimension d, irreducible and non-singular, the symmetric product $(X)^{(\Gamma)}$ has a singular locus of dimension $\leqq(q-1)d$.*

c) Let us suppose now that X is an irreducible algebraic projective variety of dimension 1 (i.e., a curve) and consider the usual symmetric product $(X)^{(q)}$. We have the following useful

PROPOSITION 2. *The symmetric product $(X)^{(q)}$ of an algebraic non-singular curve X is non-singular.*

Let $(A_1)+\cdots+(A_q)=z$ be a point of $(X)^{(q)}$ and suppose we have $A_1+\cdots+A_q=n_1P_1+\cdots+n_rP_r$ with $P_i\neq P_j$ and $\sum n_i=q$. The point z may be supposed in an affine open set of type $(U)^{(q)}$, U being an affine open set on X containing $P_1,\cdots,P_r$. Moreover, we may suppose there is a regular function t on U such that i) $t-(t)_{P_i}$ is a uniformizing parameter on X at P_i for $1\leqq i\leqq r$, ii) $(t)_{P_i}\neq(t)_{P_j}$ if $i\neq j$. Denote by t_s the regular function induced on $(U)^q$ by the projection on the s-th factor by the function t on U. Put $\phi_1=t_1+\cdots+t_q,\cdots,\phi_q=t_1\cdots t_q$; these are rational regular functions on $(U)^q$ and invariant under $\boldsymbol{G}_q$, hence rational functions

on $(X)^{(q)}$ regular at z. Let R be a rational function on $(X)^{(q)}$ regular at z. If we lift R to $(X)^q$, R will become a rational function, invariant under G_q, and regular at $A_1 \times \cdots \times A_q$ (and hence also at the $r!$ transforms of $A_1 \times \cdots \times A_q$). Now the parameters $t_1 - (t_1)_{A_1}, \cdots, t_q - (t_q)_{A_q}$ are uniformizing parameters at $A_1 \times \cdots \times A_q$. Hence we may find for each integer m a polynomial of degree m, $P_m(t_1 - (t_1)_{A_1}, \cdots, t_q - (t_q)_{A_q})$ such that $R - P_m \in \mathcal{O}_{A_1 \times \cdots \times A_q}(t_1 - (t_1)_{A_1}, \cdots, t_q - (t_q)_{A_q})^{m+1}$ ($\mathcal{O}_{A_1 \times \cdots \times A_q}$ = local ring at $A_1 \times \cdots \times A_q$). Since R is G_q-invariant, we deduce that

$$P_m(\cdots, t_i - (t_i)_{A_i}, \cdots, t_j - (t_j)_{A_j}, \cdots) = P_m(\cdots, t_j - (t_j)_{A_j}, \cdots, t_i - (t_i)_{A_i}, \cdots).$$

This property shows that P_m may be expressed as a polynomial in $\phi_1 - (\phi_1)_z, \cdots, \phi_q - (\phi_q)_z$. This last statement is an immediate consequence of our proposition for the case $X = k$. This proves the proposition in general.

2. Holomorphic forms on symmetric products. a) Let X be an abstract variety, U an open subset of X. An expression of the type $\omega^{(k)} = \sum X_{i_1 \cdots i_k} dY_{i_1} \cdots dY_{i_k}$, where the X's and Y's are rational functions on U, will be said to be a meromorphic k-form on U. With the obvious restriction of an open set U on an open subset $V \subset U$, we get the definition of the sheaf $\mathfrak{M}^{(k)}$ of meromorphic k-forms on X. If X is an irreducible variety of dimension d and $k(X)$ is the field of rational functions on X, the space $H^0(X, \mathfrak{M}^{(k)})$ is a vector space over $k(X)$ of dimension $\binom{d}{k}$. Any system $x_1, \cdots, x_d$ of elements of $k(X)$ which give a separating transcendency base for $k(X)$ generates, by the differentials $dx_1, \cdots, dx_d$, the whole space of meromorphic k-forms.

b) Let P be a point of X. The form $\omega^{(k)}$ will be said to be holomorphic at P if it has an expression in which the X's and Y's are regular at P. In particular, if P is a simple point and $t_1, \cdots, t_d$ are uniformizing parameters at P, a holomorphic form $\omega^{(k)}$ at P may be written in a *unique* way as $\sum \phi_{i_1 \cdots i_k}(t_1, \cdots, t_d) dt_{i_1} \wedge \cdots \wedge dt_{i_k}$, where the $\phi_{i_1 \cdots i_k}$ are regular at P and expressible as formal power series in the t's. A k-form holomorphic at each point of the open set U will be said to be holomorphic in U. The sheaf of holomorphic k-forms on X will be denoted by $\Omega^{(k)}$. As well as the holomorphic k-forms on X, i.e. the elements of the space $H^0(X, \Omega^{(k)})$, one may consider the holomorphic tensor forms on X, i.e. the elements of spaces of the type $H^0(X, \Omega^{(k_1)} \otimes \cdots \otimes \Omega^{(k_r)})$, the tensor product being taken over the sheaf of local rings.

c) We want to compute the dimension of some of the above mentioned vector spaces for the case in which X is the symmetric product of a projective curve. The easiest way to do it is to give first the computation for the cartesian product and then deduce from it the calculus of the vector spaces being considered.

For a cartesian product, the computation is a straightforward conseqence of the Künneth formula that will be recalled here.

LEMMA.[6] *Let X, Y be two non-singular irreducible abstract varieties. There exist coverings U_i of X, V_j of Y with affine open sets such that on the covering W_{ij} of $X \times Y$, where $W_{ij} = U_i \times V_j$, we have*

$$H^0(W_{ij}; \Omega^{(k_1)} \otimes \cdots \otimes \Omega^{(k_r)}) \approx \sum_{\substack{p_h+q_h=k_h \\ 1 \leqq h \leqq r}} H^0(U_i; \Omega^{(p_1)} \otimes \cdots \otimes \Omega^{(p_r)}) \otimes H^0(V_j; \Omega^{(q_1)} \otimes \cdots \otimes \Omega^{(q_r)}).$$

The lemma being of a local nature, we may assume X, Y to be affine varieties. Let P be a point in X, and denote by $X_1, \cdots, X_d$, $d = \dim X$, d rational functions on X, regular at P and such that the functions $t_i = X_i - X_i(P)$ are uniformizing parameters at P. Now there exists an open neighborhood U_P of P such that for any point $Q \in U_P$, the functions $X_i - X_i(Q)$ are uniformizing parameters at Q. Hence we have a covering of X, $\{U_P\}_{P \in X}$. Similarly, we define the covering $\{V_Q\}_{Q \in Y}$ of Y. Refining, if necessary, these coverings, we may suppose that the U_i (V_j) are affine varieties so that we are reduced to proving the lemma for two affine varieties X, Y which may be covered by a global system of uniformizing parameters: X_i, $1 \leqq i \leqq \dim X$; Y_j, $1 \leqq j \leqq \dim Y$. At this stage, the proof becomes evident since the ring of coordinates on $X \times Y$ is the tensor product of the rings of coordinates on X and Y.

The lemma shows that the cohomology of $X \times Y$ with coefficients in the sheaf $\Omega^{(k_1)} \otimes \cdots \otimes \Omega^{(k_r)}$ may be computed as the cohomology of the complex which appears on the right hand side of the above formula. This last is a sum of cartesian products of complexes defined for the spaces X, Y respectively. Applying now the Eilenberg-Zilber theorem [2] we get the *Künneth formula*:[7]

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[6] For a similar case, see J. P. Serre [7, (b)], Ch. VII, n. 5.

[7] We get an analogous formula if we consider instead of the sheaf of cotangent vectors and its exterior algebra, the sheaf of tangent vectors.

$$H^q(X\times Y;\Omega^{(k_1)}\otimes\cdots\otimes\Omega^{(k_r)})$$
$$=\sum_{\substack{p_h+q_h=k_h\\ 1\leqq h\leqq r}}\sum_{a+b=q}H^a(X;\Omega^{(p_1)}\otimes\cdots\otimes\Omega^{(p_r)})\otimes H^b(Y;\Omega^{(q_1)}\otimes\cdots\otimes\Omega^{(q_r)}).$$

3. a) We will apply the Künneth formula to the simple case in which $k_1=k_2=\cdots=k_r=\dim X+\dim Y$ to get the plurigenera of the product $X\times Y$. In this case, let us put $\mathcal{H}_r^*(X)=\sum_{q\geqq 0}H^q(X,\underbrace{\Omega^{(d)}\otimes\cdots\otimes\Omega^{(d)}}_{r\text{ times}})$, where $d=\dim X$, and similarly define the groups $\mathcal{H}_r^*(Y)$ and $\mathcal{H}_r^*(X\times Y)$. Then the Künneth formula reduces to

$$\mathcal{H}_r^*(X\times Y)=\mathcal{H}_r^*(X)\otimes\mathcal{H}_r^*(Y).$$

In particular, if we denote by $P_r(X)$ the $\dim_k H^0(X,\underbrace{\Omega^{(d)}\otimes\cdots\otimes\Omega^{(d)}}_{r\text{ times}})$ (r-genus of X), we get the formula $P_r(X\times Y)=P_r(X)P_r(Y)$.

Suppose that X is an algebraic projective irreducible curve. For the cartesian product $(X)^q$, we get the following formulae:

$$P_1((X)^q)=g^n;P_r((X^q)=(2r-1)^q(g-1)^q,r\geqq 2,$$

where g is the genus of the curve X.

b) Consider now the symmetric product $(X)^{(q)}$ of the curve X. Taking into consideration the regular map $\pi\colon (X)^q\to(X)^{(q)}$, any holomorphic form on $(X)^{(q)}$ may be lifted to $(X)^q$ and there it will be an element of the space of holomorphic forms of that type invariant under the symmetric group G_q.

In particular, for the plurigenera, we will have the inequality

$$P_r((X)^{(q)})\leqq\dim\{H^0((X)^q,\underbrace{\Omega^{(q)}\otimes\cdots\otimes\Omega^{(q)}}_{r\text{ times}})\}^{G_q},$$

where the symbol $\{\ \}^{G_q}$ indicates the subspace of fixed elements under G_q of the space in the brackets.

c) For the case $r=1$ (geometric genus), we have the following:

PROPOSITION 3. i) *Denote by* $\omega_\alpha(x)$, $1\leqq\alpha\leqq g$, *a base for* $H^0(X,\Omega^1)$. *Any* q*-form* $\Omega\in H^0((X)^q,\Omega^{(q)})$ *is of type*

$$\Omega=\sum c_{\alpha_1\cdots\alpha_q}\omega_{\alpha_1}(x_1)\wedge\cdots\wedge\omega_{\alpha_q}(x_q),$$

where $1\leqq\alpha_i\leqq g$ *and* $c_{\alpha_1\cdots\alpha_q}\in k$.

ii) *If the characteristic p of the field k is not 2, then any q-form $\Omega \in \{H^0((X^q, \Omega^{(q)})\}^{G_q}$ is of type*

$$\Omega = \sum_{\alpha_1 < \cdots < \alpha_q} c_{\alpha_1 \cdots \alpha_q} \operatorname{Det.} \begin{pmatrix} \omega_{\alpha_1}(x_1), \cdots, \omega_{\alpha_q}(x_1) \\ \cdots\cdots\cdots\cdots \\ \omega_{\alpha_1}(x_q), \cdots, \omega_{\alpha_q}(x_q) \end{pmatrix}.$$

If $p = 2$, then a set of independent generators for the G_q-invariant q-forms is given by the expressions $\sum \sigma_i(\omega_{\alpha_1}(x_1) \wedge \cdots \wedge \omega_{\alpha_q}(x_q))$, where σ_i runs over a system of representatives of $G_q/h_{(\alpha_1 \cdots \alpha_q)}$, $h_{(\alpha_1 \cdots \alpha_q)}$ being the subgroup of G_q which leaves fixed the permutation $(\alpha_1 \cdots \alpha_q)$.

iii) *For any value of the characteristic p, if $\pi: (X)^q \to (X)^{(q)}$ is the natural map, then any form $\Omega \in \pi^* H^0((X)^{(q)}, \Omega^{(q)})$ is of type:*

$$\Omega = \sum_{\alpha_1 < \cdots < \alpha_q} c_{\alpha_1 \cdots \alpha_q} \operatorname{Det.} \begin{pmatrix} \omega_{\alpha_1}(x_1), \cdots, \omega_{\alpha_q}(x_1) \\ \cdots\cdots\cdots\cdots \\ \omega_{\alpha_1}(x_q), \cdots, \omega_{\alpha_q}(x_q) \end{pmatrix}.$$

The points i) and ii) are obvious consequences of the Künneth formula and of a self-evident computation. Only point iii) deserves a little attention.

For that purpose, cover $(X)^{(q)}$ by affine open sets of the form $(U)^{(q)}$, where we take as uniformizing parameters the symmetric functions $\phi_1 = t_1 + \cdots + t_q, \cdots, \phi_q = t_1 \cdots t_q$ (the function $t_i - (t_i)_P$ being a uniformizing parameter at P on the i-th factor U of $(U)^q$) [8]. We get the expression: $d\phi_1 \wedge \cdots \wedge d\phi_q = \prod_{i<j} (t_i - t_j)\, dt_1 \wedge \cdots \wedge dt_q$. Now obviously

$$\pi^* H^0((X)^{(q)}, \Omega^q) \subset \{H^0((X)^q, \Omega^{(q)})\}^{G_q},$$

and the necessary and sufficient condition for a G_q-invariant q-form Ω of $(X)^q$ to be the image of a form on $(X)^{(q)}$ is that on any open set $(U)^q$, $\Omega(\prod(t_i - t_j))^{-1}$ be holomorphic there. Now the only invariant elements which satisfy this condition are the forms of the type given as one verifies.

4. A digression on the invariance of holomorphic forms. The importance of holomorphic forms on an algebraic variety lies in the fact that these are birational invariants. Precisely, we have the

THEOREM. *Let X, Y be two algebraic varieties, of the same dimensions, both irreducible and* non-singular. *Let $\phi: X \to Y$ be a birational map of X into Y. Then there is a natural* [9] *isomorphism*

---

[8] Cf. Proposition 2.

[9] The isomorphism is the one induced by the isomorphism between the spaces of global meromorphic forms.

$$\phi^*: H^0(Y, \Omega^*) \to H^0(X, \Omega^*) \ (\Omega^* = \Omega^{(p_1)} \otimes \cdots \otimes \Omega^{(p_r)}).$$

This theorem has been proved for the abstract case by Koizumi [5]. Another proof has been announced by Cartier [1]. We give here a proof close to the classical one.[10]

LEMMA 1. *The mapping ϕ^{-1} is regular on Y outside an algebraic subset of dimension $\leqq d-2$.*

Since X is complete, $\phi(X)$ (i.e. the projection on Y of the closed graph of ϕ on $X \times Y$) is closed in Y; since ϕ is birational, $\phi(X)$ contains an open subset of Y; since Y is irreducible $\phi(X) = Y$. Hence it is enough to prove the lemma for the case in which Y is an affine variety, X an algebraic variety and $\phi: X \to Y$ a regular map of X onto Y: $\phi(X) = Y$.

Let $\{U_i\}$ be an open covering of X of affine varieties U_i; denote by x_{U_i} the coordinates on U_i and y the coordinates on Y. The birational map is then given by equations $y = P_{U_i}(x_{U_i})$, $x_{U_i} = Q_{U_i}(y)$, where the P_{U_i} are polynomials in the x_{U_i}'s and the Q_{U_i} are rational functions of y. Let $x_0 \in U_i$ be a point of X; if in $y_0 = \phi(x_0)$, each Q_{U_i} or its inverse is regular, then ϕ^{-1} is regular at y_0. In fact, we have the identities $Q_{U_i,a}(P_{U_i}(x_{U_i})) = x_{U_i,a}$ as rational functions in $k(U_i)$; now if the Q_{U_i}'s are regular at y_0, ϕ^{-1} is obviously regular there. If, say, $Q_{U_i,a}$ is not regular at y_0, but $Q_{U_i,a}^{-1} = R$ is regular at y_0, we must have $R(y_0) = 0$. Hence in the identity $x_{U_i,a} R(P_{U_i}(x_{U_i})) = 1$, specializing $x_{U_i} \to x_0$, we get a contradiction. Therefore the above assertion is proved.

The lemma will be proved if we prove

LEMMA 2. *Let V be an affine irreducible variety of dimension d whose singular set is of dimension $\leqq d-2$, and let R be a rational non-zero function on V. The locus where R and R^{-1} are not regular is an algebraic subvariety of dimension $\leqq d-2$.*

If R is regular at a point, it is regular in an open neighborhood of that point. Hence the set where R is not regular is a closed set F_R; $F_R \cap F_{R^{-1}} = W$ is also closed, hence the locus W is algebraic. If $\dim W = d-1$, one may find a point $P \in V$ such that

i) P is simple on V, ii) P belongs to an irreducible component of dimension $d-1$ of W, say W_1.

At P, the local ring $\mathcal{O}_P$ on V is a unique factorization domain (since this

[10] See [9, (a)], Ch. V, Theorems 8 and 13.

is so for its completion). Hence the variety W_1 is represented in $\mathcal{O}_P$ by a prime ideal $\mathfrak{p}_1$ of dimension $d-1$, and thus is principal: $\mathfrak{p}_1 = \mathcal{O}_P\pi$. We may suppose π represented by a polynomial.

Now let R be of the form p/q, where p, q are polynomials. In $\mathcal{O}_P$, $p = \pi^\alpha p_1$, $q = \pi^\beta q_1$ with p_1, q_1 not in $\mathfrak{p}_1$. We have $R = \pi^{\alpha-\beta} p_1/q_1$. Suppose, for instance, $\alpha \geqq \beta$; then R is regular where p_1, q_1 are regular and $q_1 \neq 0$. Now there is at least one point on W_1 where p_1, q_1 are regular and $q_1 \neq 0$. In fact, p_1, q_1 are regular at P, hence in an open set on W_1. If $q_1 = 0$ contains W_1, by Hilbert's nullstellensatz, we have $q_1 \in \mathfrak{p}_1$. This is a contradiction.

To get the theorem, we have only to make use of a covering of X, Y of the type used in the lemma of n. 2 and combine lemma 1 with the following proposition, which is an abstract form of a theorem of Hartogs.

PROPOSITION 4. *Let V be an irreducible affine* normal *variety of dimension d; let W be a subvariety of V of dimension $\leqq d-2$ and let t be a rational regular function defined on $V-W$. The function t may be extended in a unique way to all V by a rational regular function.*

Denote by $\mathfrak{p}(W)$ the ideal defining the variety W in the ring $I(V)$ of coordinates on V. Let $a \in \mathfrak{p}(W)$, $a \neq 0$, and consider the ideal $I(V)a$. Since V is normal, this ideal $I(V)a$ is pure of dimension $d-1$, i.e. we have a primary decomposition $I(V)a = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_h$ with the $\mathfrak{q}_i$'s primary and associated to prime ideals $\mathfrak{p}_i$ of dimension $d-1$. Now we can find an element $b \in \mathfrak{p}(W)$ such that a and b are relatively prime (i.e. if $b \mid ac$, then $b \mid c$). In fact, denote by α_i a zero of $\mathfrak{p}_i$ which is not a zero of $\mathfrak{p}(W)$ and by l the maximum degree of the polynomials of a base of $\mathfrak{p}(W)$ as an ideal in $k[x]$. Since W is the locus of common zeros for the polynomials of $\mathfrak{p}(W)$ of degree $\leqq l$, the polynomials of $\mathfrak{p}(W)$ of degree $\leqq l$ vanishing on α_i form a proper subspace of the vector space of polynomials of $\mathfrak{p}(W)$ of degree $\leqq l$. Hence there exists a polynomial b (of degree $\leqq l$) in $\mathfrak{p}(W)$ such that $b(\alpha_i) \neq 0$ for $1 \leqq i \leqq h$, i.e. $b \notin \mathfrak{p}_i$, and this implies that b is prime to a.

Now $I(V)(a, b) \subset \mathfrak{p}(W)$ and has dimension $\leqq d-2$. We may suppose, without loss of generality, that W is the locus $a = b = 0$.

By hypothesis, t is an element of $\Gamma(V-W, \mathcal{O} \mid V-W)$, where $\mathcal{O}$ is the sheaf of local rings on V, hence for each point $P \in V-W$ we have an expression $t = r_P/s_P$ with r_P, s_P polynomials of $k[x]$ and $s_P(P) \neq 0$.

Since V is irreducible, the open sets $s_P \neq 0$, $s_Q \neq 0$ intersect in an open set on which $r_P s_Q = r_Q s_P$ (outside W), hence this equality holds as an identity in $I(V)$. Moreover, the functions s_P cannot have common zeros outside W.

Therefore a and b vanish on the zeros of the ideal generated by s_P, $P \in V - W$. By Hilbert's nullstellensatz, we can find a suitable integer ρ such that the following relations are satisfied: $a^\rho = \sum a_P s_P$, $b^\rho = \sum b_P s_P$ with the a_P's, b_P's almost all zeros.

Consider the two rational functions

$$t_a = (1/a^\rho)\sum a_P r_P, \qquad t_b = (1/b^\rho)\sum b_P r_P$$

on the open set $a \neq 0$. We have $t - t_a = 0$; similarly, on $b \neq 0$, we have $t = t_b$. Therefore on $a \neq 0$, $b \neq 0$, we have $t_a = t_b$ and hence the identity in $I(V)$: $a^\rho \sum b_P r_P = b^\rho \sum a_P r_P$. Since b is prime to a, we get $b^\rho \mid \sum b_P r_P$, i.e. $\sum b_P r_P = b^\rho \theta$ with $\theta \in I(V)$. Hence $t = \theta$, so t can be extended by θ to all V. The extension is unique since two such functions coincide on an open set of V.

Section 2. Canonical mappings of symmetric products.

5. Canonical mappings. a) Denote by $\omega_{\alpha_1 \cdots \alpha_q}$ the base for the holomorphic q-forms on $(X)^{(q)}$ described in Proposition 3. Assuming the $\omega_{\alpha_1 \cdots \alpha_q}$ as homogeneous coordinates on a projective space $P_{d(q)}(k)$, we are led to consider the mapping $\phi^{(q)}$ given by $x \to \omega_{\alpha_1 \cdots \alpha_q}(x)$ of $X^{(q)}$ into $P_{d(q)}(k)$.[11]

Since $d(q) = \binom{g}{q} - 1$, those mappings have interest only if $1 \leqq q \leqq g - 1$. The cases $q = 1$, $q = g - 1$ will have particular interest.

b) *The case* $q = 1$. A curve X of genus $g > 1$ is said to be hyperelliptic if it contains a linear series g^1_2 of degree 2 and (projective) dimension 1. This is equivalent to saying that the field $k(X)$ is a quadratic extension of a field of genus zero. We have the following:

PROPOSITION 5. *Suppose that X is of genus $g > 1$. The image $C = \phi(X)$ of X by the canonical map is non-singular; if X is not hyperelliptic, then $\phi: X \to C$ is biregular; if X is hyperelliptic, then the canonical map may be factored into the map: $X \to P_1(k)$ given by the subfield of genus 0 of $k(X)$ and the biregular map of $P_1(k)$ onto a rational normal curve*[12] *of $P_{g-1}(k)$.*

If two places A, B of X have the same center on C, the residual series, with respect to the canonical series, of $A + B$ will be of degree $2g - 4$ and

[11] These are rational maps, not necessarily regular.

[12] I. e. a projective transform of the curve with the general point $(1, \lambda, \lambda^2, \cdots, \lambda^{g-1})$.

dimension $g-2$. By the theorem of Riemann-Roch (in the form of the reciprocity theorem), the dimension x of $|A+B|$ will be given by

$$(2g-4)-2=2(g-2)-x.$$

Hence $x=1$ and X is hyperelliptic. This shows that the canonical map ϕ is one-to-one if X is not hyperelliptic.

Assume now that X is not hyperelliptic, and let us suppose that ϕ is not biregular at a point $A \in X$. Denote by $\mathcal{O}_A$, $\mathcal{O}_{\phi(A)}$ the local rings of A, $\phi(A)$ respectively on X, C. The map ϕ induces a monomorphism $\phi^*: \mathcal{O}_{\phi(A)} \to \mathcal{O}_A$ since ϕ is one-to-one. Moreover $\phi^*\mathcal{O}_{\phi(A)} \subsetneq \mathcal{O}_A$ since ϕ is not biregular at A.

Denote by y_i non-homogeneous coordinates on C, $\phi(A)$ being supposed at the origin. We must have $v_A(\phi^* y_i) \geqq 2$, hence, for any hyperplane $\sum a_i y_i = 0$ containing $\phi(A)$, we get $v_A(\phi^* \sum a_i y_i) \geqq 2$. This shows again that the linear series $|2A|$ will have dimension 1, contrary to the fact that X is not hyperelliptic.

Hence if X is not hyperelliptic, it follows that $\phi: X \to C$ is a biregular map, and C is therefore non-singular.

If X is hyperelliptic, we have only to recall that the canonical series is compounded with the g^1_2 and that therefore it is the image on X, by ϕ^{-1}, of the series of all divisors of degree $g-1$ on $P_1(k)$.

c) *The general case.* If x is a general point on X, $\omega_1(x), \cdots, \omega_g(x)$ are the homogeneous coordinates of a general point on the canonical model C of X. If $x_1, \cdots, x_q$ are q independent general points on X, then

$$\omega_{\alpha_1 \cdots \alpha_q} = \text{Det.} \begin{pmatrix} \omega_{\alpha_1}(x_1), \cdots, \omega_{\alpha_q}(x_1) \\ \cdots\cdots\cdots\cdots \\ \omega_{\alpha_1}(x_q), \cdots, \omega_{\alpha_q}(x_q) \end{pmatrix}$$

are the Plucker's coordinates of the projective space $P_{q-1}(k)$ spanned by the q points $(\omega_1(x_i), \cdots, \omega_g(x_i))$, $1 \leqq i \leqq q$. (These are linearly independent since C belongs to $P_{g-1}(k)$.) We deduce the following *geometric construction of the canonical map*:

The canonical model $C^{(q)}$ of the symmetric product $(X)^{(q)}$, $1 \leqq q \leqq g-1$, (g = genus of the curve X) has for general point the point of the Grassmann manifold $G_{q-1, g-q}$ (of the spaces P_{q-1} in P_{g-1}) which is the image of the space P spanned by q general points of the canonical curve C.

In particular, for $q = g-1$, the canonical model $C^{(g-1)}$ of $(X)^{(g-1)}$ is the projective space $P_{g-1}{}^*(k)$ dual to the space $P_{g-1}(k)$ of the canonical curve C.

d) The canonical maps $\phi^{(q)}$ are rational maps but not necessarily regular maps (if $q>1$).

PROPOSITION 6. *Let* $(A_1)+\cdots+(A_q)$ *be the point of* $(X)^{(q)}$ *corresponding to the divisor* $A_1+\cdots+A_q$ *on* X. *Then* $\dim|A_1+\cdots+A_q|\geqq 1$ *if and only if for any holomorphic q-form* Ω *on* $(X)^{(q)}$, *we have* $\Omega((A_1)+\cdots+(A_q))=0$ (*i.e. the point* $(A_1)+\cdots+(A_q)$ *is a base point of the canonical system*).

Let $A_1+\cdots+A_q=n_1P_1+\cdots+n_rP_r$ with $P_i\neq P_j$ and $\sum n_i=q$. Let ω_α, $1\leqq\alpha\leqq g$, be a base for holomorphic 1-forms on X. Choose a local parameter t_P at the point P on X and write

$$\omega_\alpha=\{\phi_\alpha(P)+\phi_\alpha'(P)t_P+\phi_\alpha''(P)t_P^2+\cdots\}dt_P.$$

The condition $\dim|A_1+\cdots+A_q|>1$ means that the specialty index $i(A_1+\cdots+A_q)>g-q$. Now $i(A_1+\cdots+A_q)=s$ means:

$$\operatorname{rank}(\phi(P_1),\cdots,\phi^{(n_1-1)}(P_1);\cdots;\phi(P_r),\cdots,\phi^{(n_r-1)}(P_r))=g-s,$$

where the symbol $\phi^{(s)}(P_i)$ stands for the column vector $\phi_\alpha{}^{(s)}(P_i)$.

Using the notation of Proposition 2, if $r=1$, for the form

$$\Omega_{\alpha_1\cdots\alpha_q}=\{(\sum\omega_{\alpha_1}(Q_i))\wedge\cdots\wedge(\sum\omega_{\alpha_q}(Q_i))\}\{\prod((t)_{Q_i}-(t)_{Q_j})\}^{-1},$$

we get the expression at P_1:[13]

$$\Omega_{\alpha_1\cdots\alpha_p}(P_1)=\operatorname{Det}.\begin{pmatrix}\phi_{\alpha_1}(P_1),\cdots,\phi_{\alpha_1}{}^{(n_1-1)}(P_1)\\ \cdots\cdots\cdots\\ \phi_{\alpha_q}(P_1),\cdots,\phi_{\alpha_q}{}^{(n_1-1)}(P_1)\end{pmatrix}\cdot d\phi_1\wedge\cdots\wedge d\phi_{n_1}\qquad(n_1=q).$$

In the general case, we get therefore

$$\Omega_{\alpha_1\cdots\alpha_q}(n_1(P_1)+\cdots+n_r(P_r))$$
$$=\{\prod_{i<j}((t)_{P_i}-(t)_{P_j})\}^{-1}\sum\Omega_{a_1\cdots a_{n_1}}(P_1)\wedge\cdots\wedge\Omega_{c_1\cdots c_{n_r}}(P_r),$$

where $(a_1,\cdots,c_{n_r})$ is any permutation of $(\alpha_1,\cdots,\alpha_q)$ and in the product, each factor has to be taken the correct number of times. The differentials on the right side at $(A_1)+\cdots+(A_q)$ have for jacobian with respect to $d\phi_1,\cdots,d\phi_q$ the product which appears in the formula. It follows that the

[13] From the obvious identity:

$$\begin{pmatrix}\phi_{\alpha_1}(P)\cdots\\ \cdots\\ \cdots\phi_{\alpha_q}{}^{(q-1)}(P)\end{pmatrix}\begin{pmatrix}1\cdots 1\\ \theta_1\cdots\theta_q\\ \theta_1{}^{q-1}\cdots\theta_q{}^{q-1}\end{pmatrix}=\begin{pmatrix}\sum\phi_{\alpha_1}{}^{(s)}\theta_1{}^s,\cdots\\ \cdots,\sum\phi_{\alpha_q}{}^{(s)}\theta_q{}^s\end{pmatrix}$$

by taking the determinants and putting $\theta_i=t_i-(t_i)_P$.

condition $\Omega((A_1)+\cdots+(A_q))=0$ means that all minor determinants of order q of the above matrix are zero. q. e. d.

COROLLARY. *The map $\phi^{(q)}$ is regular at $(A_1)+\cdots+(A_q)$ if*

$$\dim.|A_1+\cdots+A_q|=0.^{14}$$

We finally remark that the canonical models $C^{(q)}=\phi^{(q)}((X)^{(q)})$ are intrinsically defined as projective varieties in terms of the field $K=k((X)^{(q)})$. We have in fact the following general

PROPOSITION 7. *Let K be the field of rational functions of a complete non-singular variety V of dimension d. If $\dim H^0(V,\Omega^{(d)})>0$, then we can define the canonical image C of V. The variety C is determined with its structure of projective variety by the field K.*

The space of holomorphic d-forms of K is intrinsically defined (n. 4). Let $\Omega_1,\cdots,\Omega_\rho$ be a base of this space (ρ is certainly finite since the sheaf $\Omega^{(d)}$ is coherent).[15] Taking the Ω's as "homogeneous coordinates"[16] of a general point on a projective space $P_{\rho-1}$ we get as locus in $P_{\rho-1}$ the variety C. Since the Ω's are defined modulo a linear substitution, C is defined modulo a projective transformation. The definition depends solely on K.

6. Further properties of the canonical mappings. a) We give here some properties of the canonical mappings that will be needed later.

PROPOSITION 8. *Let C be the canonical image of a non-hyperelliptic curve. Then*

(α) the tangent line t_P at the general point P of C is not tangent at any other point,

(β) the general hyperplane cuts C in $2g-2$ points such that any $g-1$ of them are linearly independent.

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[14] The image point $\phi^{(q)}((A_1)+\cdots+(A_q))$ may, however, be a singular point (for example, $g=4, q=2$).

[15] J. P. Serre [7, (a)].

[16] This means that we consider in the graded ring $k[X_1,\cdots,X_\rho]$ of homogeneous polynomials the ideal defined as the kernel of the homomorphisms given by $f(X_1,\ldots,X_\rho)\to f(\Omega_1,\ldots,\Omega_\rho)$ of $k[X_1,\cdots,X_\rho]$ into the ring of holomorphic tensor forms. The construction can also be described in the following form. We consider in K the rings $A_i=k[X_i^1,\cdots,X_i^\rho]$, where $X_i=\Omega_h/\Omega_i$, and we assume A_i as ring of coordinates of the affine variety C_i in the open set $U_i=\{y\mid y\in P_{\rho-1}, y_i\neq 0\}$ and we "put them together" by the change of coordinates $X_i^h=X_j^h/X_j^i$ to build up the variety C.

Proof of (α). Suppose t_P is tangent to C at the point $Q \neq P$. The specialty index of $2P+2Q$ is $i(2P+2Q) = g-2$, hence $\dim |2P+2Q| = 2$. We have therefore a g^2_4 on C. This series gives a map of C on a plane quartic (since we cannot have $g^2_4 = 2g^2_2$ nor can the image be a conic since C is not rational). Hence the genus of C is $\leqq 3$.[17] This proves (α) for $g > 3$.

If $g = 3$, C is then a plane non-singular quartic. If t_P touches C also at $Q \neq P$, the dual curve Γ[18] of C will be a plane curve such that the general point t_P belongs to two different lines p, q (corresponding to P and Q) which meet Γ only at t_P. This implies that Γ is a line, i.e. the tangents to C will pass through the same point O (necessarily not on C). The lines containing O will give on C a series of type $2g^1_2$. This is impossible since C is not hyperelliptic.[19]

Proof of (β). The general hyperplane u cuts C in $2g-2$ distinct points. Suppose that $g-1$ of them are linearly dependent, i.e. the space spanned by $g-2$ of them contains the remaining point. There exists, therefore, an integer h such that $P_1, \cdots, P_h$ are linearly dependent while $P_1, \cdots, P_{h-1}$ are independent. Certainly $h \geqq 3$. Let $P_{g-1}{}^*$ be the dual space of P_{g-1} (the space of C), let G be the Grassmann manifold of subspaces of P_{g-1} of dimension $h-2$, and consider the irreducible correspondence T in $P_{g-1}{}^* \times G$ with the general point (u, space determined by $P_1, \cdots, P_{h-1}$). The principle of counting the constants[20] gives

$$\dim(P_{g-1}{}^*) + 0 = \dim T = \dim \mathrm{pr}_G(T) + g - h - 2.$$

Therefore $\dim \mathrm{pr}_G(T) = h-1$. Now the variety on G whose general point represents the subspace of P_{g-1} of dimension $h-2$ spanned by $h-1$ independent general points of C has dimension $\leqq h-1$ and contains $\mathrm{pr}_G(T)$. It follows that our assumption implies that the general subspace spanned by $h-1$ independent general points contains another point of C (distinct from the given $h-1$).

Suppose that h is the minimum integer of this sort.

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[17] J. P. Serre [7, (b)], Ch. IV, n. 8.

[18] The definition of the "dual curve" will be recalled at n. 10.

[19] We can also remark that if the tangent line at a generic point P of C is also tangent at Q, the series $P+Q$ is obtained by dividing by 2 the canonical series of C. But division of a linear series leads to a finite number of different results. (A. Weil, [9, (b)], p. 117). Hence we must have at least two couples $P+Q$, $P'+Q'$ without common points and linearly equivalent. This implies the existence on C of a g^1_2.

[20] W. V. D. Hodge and D. Pedoe [4], volume 2, p. 104.

If $h=3$, then any line joining two general points of C meets C again in some other point. This implies that C has all tangent lines passing through the same point O.[21] The same will hold for a generic projection C' of C on a plane. Now C' is birationally equivalent to C.[22] The dual curve Γ of C' is a line. Moreover, the tangent line $t_{P'}$ at a general point of C' will not be tangent at any other point. This is because of α) and the fact that by generic projection the property is preserved, as we shall now prove. Consider the linear series cut out on C by the hyperplanes through a tangent line t_P; it is of dimension $g-3$ and degree $\leqq 2g-4$. The general divisor of this series cannot be obtained by taking p-times the general divisor of another linear series, because that would imply that C is hyperelliptic or rational. This shows that the property α) is maintained by the projection from a generic space lying in a hyperplane tangent to C at P but nowhere else. It appears now that $k(C')$ is a purely inseparable extension of $k(\Gamma)$ and hence that C' is rational. This is a contradiction.

If $h>3$, take the projection C_0 of C from the space spanned by $h-3$ general independent points. We get a rational map $C \to C_0$ which is generally one to one because h is assumed to be minimal. Hence $k(C)$ is purely inseparable over $k(C_0)$, and therefore C and C_0 have the same genus.[23] Now for C_0, we have the following properties: i) each line joining two different general points on C_0 meets C_0 again in some other point; ii) the tangent line at a general point of C_0 is not tangent in any other point of C_0. This enables us to apply to C_0 the previous argument.

Remarks 1. From the argument given above, it follows that for a non-hyperelliptic curve X, the canonical map $\phi^{(q)}: (X)^{(q)} \to C^{(q)}$ is rational and

[21] E. Lluis [6]. Here we give a short proof. Let C be an algebraic curve in $P_r(k)$, $r \geqq 3$, P, Q being independent general points on C. Assume that all hyperplanes containing P and Q all contain a third point O of C. The point O must be different from P and Q since the general hyperplane is not tangent to C at P or Q. Add the coordinates of O to the ground field and consider the projection C' of C from O. The general point P' of C' is the projection of both P, Q. The tangent lines t_P, t_Q at P, Q respectively lie in the plane joining O to the tangent line $t_{P'}$ to C' at P'. This implies that t_P, t_Q have a point in common. Since P, Q are general and independent on C, it follows that the tangent lines at any two simple points of C must intersect. Since these lines do not belong to the same plane, they belong to the same point.

[22] This is an obvious consequence of the theorem on primitive elements, B. van der Waerden [8], p. 132.

[23] Suppose $k(C_0) = k(C)^{p^a}$. Denote by σ the automorphism of the universal domain given by exponentiation to the p^a-th power. If x, y are generators of $k(C)$ and $f(x,y)=0$ is the equation they satisfy, $x\sigma$, $y\sigma$ are generators of $k(C)^{p^a} = k(C_0)$, and hence they satisfy the equation $f\sigma(x,y)=0$. From this the assertion follows.

generally one to one if $1 \leqq q \leqq g-2$, so that $k((X)^{(q)})$ will be purely inseparable over $k(C^{(q)})$.

2. The above proof also gives *Clifford's theorem* (strong form): If the linear series g^r_n is special, then $n \geqq 2r$, and if $n = 2r$ either the series is the canonical series or else the curve is hyperelliptic.

Any divisor G_n of g^r_n has specialty index $i(G_n) = r-n-g$ (i.e. the canonical divisor containing G_n satisfies $r-n$ linearly independent linear conditions). Since g^r_n has dimension r we must have $i(G_n) \leqq g-r$. Hence $n \geqq 2r$.

Suppose now that the curve that carries the series is the canonical image of a non-hyperelliptic curve. Any r points $P_1 + \cdots + P_r$ may be completed in a positive divisor G_n of g^r_n. If $n = 2r$, then $i(G_{2r}) = g-r$ and, therefore, any hyperplane through $P_1, \cdots, P_r$, if these are linearly independent, will contain the full divisor G_{2r}. Since we suppose $r < g-1$, the above argument applies to give the result.

3. Another consequence of the proposition is that *the canonical curve C is a projectively normal variety.*

This is obvious if X is hyperelliptic since C is a rational normal curve. Suppose X non-hyperelliptic and denote by σ_k the dimension of the minimal sum $|K| + \cdots + |K| \cdot k + 1$ times, where $|K|$ is the canonical series. Let K_u be the general hyperplane section of C and A, B two disjoint sets of $g-2$ distinct points each, contained in K_u. We can find a hyperplane through A not containing any other point of K_u, and the same statement holds for B. Hence K_u gives at least $2g-3$ linearly independent conditions on the minimal sum $|K| + |K|$, i.e. $\sigma_1 - (2g-3) \geqq \sigma_0$. The same argument shows that there exists a hyperplane containing A, a hyperplane containing B and a third hyperplane containing a point P of K_u outside A and B such that the union of the three hyperplanes does not contain K_u. Hence $\sigma_2 - (2g-2) \geqq \sigma_1$, and similarly $\sigma_s - (2g-2) \geqq \sigma_{s-1}$. Since $\sigma_0 = g-1$ we get $\sigma_k \geqq 2(k+1)(g-1) - g$. By the theorem of R. R., we infer that the equality must hold.

b) Consider now in $(X)^{(g-1)} \times P_{g-1}{}^*$ the closed graph Φ of the canonical mapping, the space $P_{g-1}{}^*$ being identified with the dual of the space P_{g-1} of the canonical image C of X. Denote by u the point of $P_{g-1}{}^*$ which corresponds to the hyperplane $u = 0$ in P_{g-1}.

PROPOSITION 9. *Consider for $u \in P_{g-1}{}^*$ the set of corresponding points on $(X)^{(g-1)}$ (i.e. $\mathrm{pr}_{(X)^{(g-1)}}(\mathrm{pr}^{-1}_{P_{g-1}{}^*}(u) \cap \Phi)$).*

i) *If X is not hyperelliptic, this set consists of the points on $(X)^{(g-1)}$ corresponding to positive divisors of degree $g-1$ contained in the divisor cut out on C by the hyperplane $u=0$.*

ii) *If X is hyperelliptic this set corresponds to positive divisors on X which, by the canonical map: $X \to C$, are mapped onto the divisor cut out on C by the hyperplane $u=0$.*

Proof. i). For u a general point on ${P_{g-1}}^*$, the assertion follows from Proposition 8, (β). For any u, it follows from this by specialization. Here is the formal argument. Denote by $x^{(1)}, \cdots, x^{(g-1)}$ $g-1$ general independent points on P_{g-1}. Consider the projective variety W whose general point $z=z(x^{(1)}, \cdots, x^{(g-1)})$ has the homogeneous coordinates given by the coefficients of the polynomial in $v=(v^0, \cdots, v^{g-1})$, $\prod_{i=1}^{g-1}(x_0^{(i)}v^0+\cdots+x_{g-1}^{(i)}v^{g-1})$. The variety W is a regular transform of the symmetric product $(P_{g-1})^{(g-1)}$, the mapping being strictly one to one. There exists therefore a regular and one to one mapping of $(X)^{(g-1)}$ into W. For our purposes, we may identify $(X)^{(g-1)} \times {P_{g-1}}^*$ with the corresponding image in $W \times {P_{g-1}}^*$.

Denote by $A_h=(a_0^{(h)}, \cdots, a_{g-1}^{(h)})$, $1 \leqq h \leqq 2g-2$, the points of the intersection of the general hyperplane $u=0$ with the canonical curve C. Consider the subvariety Ψ of $W \times {P_{g-1}}^*$ with general point $(z(A_1, \cdots, A_{g-1}), u)$. Since Φ is irreducible and since, by proposition 8, the points $((A_{i_1})+\cdots+(A_{i_{g-1}}), u)$ all belong to Φ, it follows that the subvariety Ψ coincides with the image of Φ. Hence Ψ contains the points $(z(A_{i_1}, \cdots, A_{i_{g-1}}), u)$ $((i_1, \cdots, i_{g-1})$ being a combination of the indices $1, \cdots, 2g-2)$. Also those points are the only points with the second projection u.

Now by specialization $u \to u_0$, the points $A_1, \cdots, A_{2g-2}$ specialize in the points of the divisor $u_0=0$ on C. This specialization gives a unique specialization for the points considered on Ψ. Since Ψ is irreducible, any point of Ψ is a specialization of a general point and therefore the specialized points are the only point of Ψ with second projection u_0.

ii) The argument is the same. The only thing to be observed is that the correspondence defined for a general u in the statement is an irreducible correspondence. To do this, consider the regular natural map p: $(X)^{(g-1)} \to (C)^{(g-1)}$, and in $(C)^{(g-1)} \times {P_{g-1}}^*$, consider the irreducible correspondence T with general point $((A_1)+\cdots+(A_{g-1}), u)$, where $A_1+\cdots+A_{g-1}$ is the divisor $u=0$ on C. The correspondence Φ is irreducible and contains the subvariety of $(X)^{(g-1)} \times {P_{g-1}}^*$ which is mapped by $(p, 1)$ onto T. This proves our assertion.

COROLLARY 1. *Let $\hat{\Phi}$ be the normalized model of the graph Φ. Then Φ is biregularly equivalent to the normalization of the canonical model $P_{g-1}{}^*$ in the field $k((X)^{(g-1)})$.*

This is an immediate consequence of Zariski's main theorem [24] since

α) $\hat{\Phi}$ is birationally equivalent to $(X)^{(g-1)}$ and locally normal,

β) $\hat{\Phi}$ is a projective variety and the projection $p: \hat{\Phi} \to P_{g-1}{}^*$ is regular,

γ) for any point $u \in P_{g-1}{}^*$, the inverse image $p^{-1}(u)$ is finite.

COROLLARY 2. *The canonical map $\phi^{(g-1)}: (X)^{(g-1)} \to P_{g-1}{}^*$ is regular at the point $(A_1) + \cdots + (A_{g-1})$ if and only if $\dim |A_1 + \cdots + A_{g-1}| = 0$ when X is not hyperelliptic. It is everywhere regular if X is hyperelliptic.*

Suppose X is not hyperelliptic. If $\dim |A_1 + \cdots + A_{g-1}| > 0$, all points $((A_1) + \cdots + (A_{g-1}), u)$, where $u = 0$ runs through all hyperplanes containing $A_1 + \cdots + A_{g-1}$, belong to the closed graph Φ of $\phi^{(g-1)}$. This shows that the set $((A_1) + \cdots + (A_{g-1})) \times P_{g-1}{}^* \cap \Phi$ is infinite and therefore $\phi^{(g-1)}$ cannot be regular at $(A_1) + \cdots + (A_{g-1})$.

Suppose X is hyperelliptic. The canonical image C of X is a rational normal curve with the general point, say, $(1, t, t^2, \cdots, t^{g-1})$. The canonical map $\phi^{(g-1)}$ may be factored into the regular natural map $(X)^{(g-1)} \to (C)^{(g-1)}$ and the map $(C)^{(g-1)} \to P_{g-1}{}^*$ given by

$$\begin{cases} u_h = \text{Det.} \begin{pmatrix} 1, t_1, \cdots, t_1{}^{h-1}, t_1{}^{h+1}, \cdots, t_1{}^{g-1} \\ \cdots\cdots\cdots\cdots\cdots \\ 1, t_{g-1}, \cdots, t_{g-1}{}^{h-1}, t_{g-1}{}^{h+1}, \cdots, t_{g-1}{}^{g-1} \end{pmatrix} \\ 0 \leqq h \leqq g-1 \end{cases}$$

which associates to the general point $(t_1) + \cdots + (t_{g-1})$ on $C^{(g-1)}$ the hyperplane containing the points $t_1, \cdots, t_{g-1}$ of C.

This second map is also regular. Computing the determinant, we find the value $\phi_{g-h-1}(t_1, \cdots, t_{g-1}) W(t_1, \cdots, t_{g-1})$,[25] where ϕ_α is the elementary symmetric function of degree α and W the Vandermonde determinant in the quantities $t_1, \cdots, t_{g-1}$. The mapping may thus be written as

$$u_h = \phi_{g-h-1}(t_1, \cdots, t_{g-1}), \qquad 0 \leqq h \leqq g-1,$$

and this proves the assumption.

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[24] O. Zariski [10], p. 82.

[25] The determinant vanishes if W vanishes. Also, it vanishes if ϕ_{g-h-1} vanishes since this is the condition that $t_1, \cdots, t_{g-1}$ be the roots of an equation of degree $g-1$, $a_{g-1}x^{g-1} + \cdots + a_{h-1}x^{h-1} + a_{h+1}x^{h+1} + \cdots + a_0 = 0$. Hence it is divisible by $\phi_{g-h-1}W$ and hence equals this product.

Section 3. The theorem of Torelli.

7. The branch locus of a rational map. a) Let $f: X \to P$ be a rational map of a projective irreducible variety of dimension d on a projective space P_d, i.e. $f^*: k(P) \to k(X)$ is a monomorphism and hence $k(X)$ is an algebraic extension of $k(P)$. We will suppose the extension a separable one, i.e. that the map f is separable.

Denote by Φ the closed graph of f, that is, the projective subvariety of $X \times P$ with general point $(x, f(x))$, where x is general point on X. We have $\mathrm{pr}_X(\Phi) = X$, $\mathrm{pr}_P(\Phi) = P$, since X and P are complete. Given a point $y \in P$, $f^{-1}(y)$ will denote the set of points $\mathrm{pr}_X(\Phi \cap X \times y)$.

b) LEMMA 1. *The points $y \in P$ such that $f^{-1}(y)$ is not finite are contained in an algebraic variety of dimension $\leqq d-2$.*[26]

Let $(y_0, \cdots, y_d)$ be homogeneous coordinates on P, denote by U_i the open set $y_i \neq 0$ and identify it with k^d where $\xi_1, \cdots, \xi_d$ are coordinates. Let V_j be a finite affine open covering of X and let $(z_1, \cdots, z_r)$ be the coordinates on V_j. The z_i's satisfy algebraic equations $a_0^{(i)}(\xi) z_i^{n_i} + \cdots + a_{n_i}^{(i)}(\xi) = 0$, where the a's may be supposed in $k[\xi]$ and without common factors. If ξ does not satisfy the equations $a_k^{(i)}(\xi) = 0$, then $f^{-1}(\xi) \cap V_j$ is finite. The locus W_{ij} defined by the equations $a_k^{(i)}(\xi) = 0$ is of dimension $\leqq d-2$. Therefore, $f^{-1}(y)$ will be finite if $y \notin \cup \bar{W}_{ij}$.

LEMMA 2. *There are points $y \in P$ such that $f^{-1}(y)$ is finite and consists of $n = [k(X) : k(P)]$ different points. These points y cover an open set in P.*

Let V be an affine open set on X with coordinates $z_1, \cdots, z_r$. We may suppose that each z_i, as an element of $k(X)$, is such that $k(P)(z_i) = k(X)$. With the same notation as above, we have irreducible equations

$$F_i \equiv a_0^{(i)}(\xi) z_i^n + \cdots + a_n^{(i)}(\xi) = 0$$

of the same degree $n = [k(X) : k(P)]$. Moreover, since $k(\xi, z_i) = k(\xi, z_j)$, we have rational expressions of the type: $z_i = R_{ij}(\xi, z_j) = P_{ij}(\xi, z_j)/Q_{ij}(\xi, z_j)$, where P, Q are polynomials in their arguments, and where Q_{ij} is prime to F_j. Hence the above expression may be written in the form $z_i = P_{ij}(\xi, z_j)/Q_{ij}(\xi)$. Now suppose that $y \in U$ does not satisfy any one of the equations $\Pi Q_{ij}(\xi) = 0$,

[26] The lemma holds for any rational map $f: X \to Y$ between algebraic manifolds of the same dimension d (there always exists a regular map $\phi: Y \to P_d$. Then apply the lemma to $\phi \circ f$).

$\Pi a_0^{(i)}(\xi) = 0$, Discriminant $(F_i) = 0$. It follows that the equations F_i have n distinct roots, and because of the above relations, there are n distinct points that correspond to y on V. Since any finite set of points can be covered with an open affine set the assertion follows.

Let A be the set of points $y \in P$ such that $f^{-1}(y)$ consists of n distinct points. The minimal algebraic variety containing the complement of A (i.e. the closure of the complement of A) will be denoted by D.

LEMMA 3. *Let X, X' be two projective and locally normal varieties birationally equivalent; hence $k(P) \subset k(X) = k(X')$. The loci D, D' relative to the corresponding rational maps $f: X \to P$, $f': X' \to P$ differ at most on a variety of dimension $\leqq d-2$.*

Going to the graph of the birational map $\phi: X \to X'$, we may suppose that ϕ is regular (we have to take the normalized model of the graph). Now since X, X' are locally normal, ϕ is a one to one map outside of a subvariety $W' \subset X'$ of dimension $\leqq d-2$ (n. 4, Lemma 1). Since the definition of D (by means of the sets $f^{-1}(y)$, $y \in P$) involves only the graph of the mapping f, we may as well suppose f (and f') regular. From this, the assertion follows.

The pure part of dimension $d-1$ of D has an intrinsic meaning in terms of the extension $k(P) \subset k(X)$ and the chosen model P of $k(P)$. It will be called the *branch locus* of the rational map $f: X \to P$.

Remarks. 1. The branch locus can be equally defined for a non-separable algebraic extension of $k(P)$ by substituting the given extension $k(X)$ with the maximal separable extension L of $k(P)$ in $k(X)$ and applying the above definition to L.

2. The branch locus can also be described in terms of valuations: it is the divisor corresponding to the valuations of the ring of coordinates of the affine model $U_i \subset P$ which are ramified in the extension field $k(X)$ of $k(U_i)$. $0 \leqq i \leqq d$.

8. The branch locus for the canonical map $f: (X)^{(p-1)} \to P^*_{p-1}$. We identify (as in n. 5, e)) the space $P_{p-1}{}^*$ with the dual space of the space P_{p-1} of the canonical image C of the curve X. Let D be the branch locus of the canonical map f.

PROPOSITION 10. i) *If X is not hyperelliptic, D is irreducible, and a*

general point of D is given by the general tangent hyperplane to the canonical curve.

ii) *If X is hyperelliptic, the locus D consists again of the irreducible hypersurface of the tangent hyperplanes to C and of the hyperplanes which represent the stars of hyperplanes in P_{g-1} with center at the branch points on C (for the canonical map $\phi: X \to C$).*

i) Since X is not hyperelliptic, X may be identified with the curve C whose order is $2g-2$ ($g =$ genus of X). If $y \notin D$, D being the set described in the proposition, then $f^{-1}(y)$ consists of exactly $\binom{2g-2}{g-1}$ different points; if $y \in D$, then $f^{-1}(y)$ consists of at most $\binom{2g-2}{g-1} - \binom{2g-4}{g-2}$ different points. Since $(X)^{(q)}$ is locally normal, this characterizes D as the branch locus if we prove that D is pure (in fact, irreducible) of dimension $g-2$.

ii) In an analogous way, if X is hyperelliptic, the same argument applies to the locus D described in the proposition (in this case, for $y \notin D$, $f^{-1}(y)$ consists of 2^{g-1} distinct points). The proof will be completed if we prove the following:

LEMMA. *Let C be a non-singular algebraic curve belonging to some projective space P_n with $n \geqq 2$; the hyperplanes tangent to C are represented by the points of an irreducible hypersurface in the dual space P^* of P.*

Denote by $(1, \xi_1, \cdots, \xi_n)$ a general point of C and let $(u_0, \cdots, u_n)$ be homogeneous coordinates on P^*. We may suppose $k(C)$ is a separable extension of $k(\xi_1)$. The two quantities:

$$u_0 = -\{u_1\xi_1 + \cdots + u_n\xi_n\}$$
$$= -\{u_2(\xi_2 - d\xi_2/d\xi_1)\xi_1) + \cdots + u_n(\xi_n - d\xi_n/d\xi_1)\xi_1)\}$$

$$u_1 = -\{u_2(d\xi_2/d\xi_1) + \cdots + u_n(d\xi_n/d\xi_1)\}$$

(which give the conditions for the hyperplane $(u_0, \cdots, u_n)$ to be tangent to C) are algebraically dependent over $k(u_2, \cdots, u_n)$. Hence we have an algebraic equation $F(u_0, u_1, u_2, \cdots, u_n) = 0$. This equation is unique since u_1 or u_2 is transcendental over $k(u_2, \cdots, u_n)$. Suppose, in fact, that this is not the case. Then the quantities $c_i = d\xi_i/d\xi_1$, $2 \leqq i \leqq n$, are constant because u_1 is algebraic over $k(u_2, \cdots, u_n)$. The qualities $d_i = \xi_i - c_i\xi_1$, $2 \leqq i \leqq n$, are also constant for an analogous reason. This implies that C is a line, which is excluded by hypothesis.

We may suppose that F is a polynomial in all the u's and irreducible, not only as a polynomial in u_0, u_1 but in all the variables. This polynomial F

is certainly homogeneous. In fact, if u satisfies $F(u)=0$, then λu must satisfy $F(\lambda u)=0$ for a general parameter λ; since $F(u)$ is irreducible and of the same degree in the u's as $F(\lambda u)$, we must have $F(\lambda u)=A(\lambda)F(u)$ with $A(\lambda)$ a polynomial in λ alone and in fact $=\lambda^{\text{degree of } F}$.

Hence the coordinates of the general tangent plane satisfy the equation $F(u)=0$. But C is non-singular; therefore, any tangent plane may be obtained by specialization from the general.[27] This proves the lemma.

9. Proof of Torelli's theorem for characteristic $p=0$. We recall the statement of the theorem:

THEOREM 1. *If the symmetric products $(X)^{(g-1)}$, $(Y)^{(g-1)}$ of two curves X, Y of the same genus g are birationally equivalent, then the curves X, Y are also birationally equivalent.*

From the hypothesis $k((X)^{(g-1)})=k((Y)^{(g-1)})$, it follows that the branch locus D in the canonical model $P_{g-1}{}^*$ is the same for both symmetric products. From the reducibility or not of the hypersurface D, we infer that both X, Y are hyperelliptic or not. We consider separately the two cases.

(a) *Non-hyperelliptic case.* From the description of the locus D given in the previous n. 7, it follows that the canonical curves C and Γ, of X and Y respectively, have the same set of tangent hyperplanes. The plane projections C', Γ' of C, Γ from a general projective subspace of dimension $g-4$ have therefore the same set of tangent lines. We conclude from that (since we are in characteristic $p=0$) that $C'=\Gamma'$. Since $k(X)=k(C)=k(C')$ and $k(Y)=k(\Gamma)=k(\Gamma')$, we have $k(X)=k(Y)$. q. e. d.

(b) *Hyperelliptic case.* Since the characteristic p is $\neq 2$, any hyperelliptic curve X corresponds to a quadratic extension $k(x,y)$ of $k(x)$, where $y^2=\prod_{i=1}^{2g+2}(x-\alpha_i)$ with $\alpha_i\neq\alpha_j$, and it is therefore uniquely determined by the $2g+2$ branch points $x=\alpha_i$. Moreover, the field $k(x)$ may be identified with the field of the canonical image of the given curve so that the curve X will be determined by the knowledge of its canonical image C and the position on it of the $2g+2$ branch points. We may also remark that in fact, the $2g+2$ branch points above are able to determine the curve X since they will

[27] Let $a=(a_0,\ldots,a_n)$ be a point of C; suppose for instance, $a_0\neq 0$ and that x_1 is a uniformizing parameter at a; specializing $\xi_i\to a_i/a_0$, the derivatives $d\xi_i/d\xi_1$ are specialized into finite quantities t_i; the specialized hyperplanes given by the above relations continue to satisfy $F(u)=0$.

determine automatically the unique normal rational curve C which contains them.[28]

Now again the branch locus D is the same for the two products $(X)^{(g-1)}$, $(Y)^{(g-1)}$. Moreover, D consists of an irreducible hypersurface of order >1[29] representing the tangent hyperplanes to a rational normal curve of P_{g-1} and of $2g+2$ hyperplanes. These hyperplanes determine in P_{g-1} the $2g+2$ branch points on the canonical image of X and of Y. The theorem follows from the above remarks.

Remark. If X, Y, and the given birational map $(X)^{(g-1)} \to (Y)^{(g-1)}$ are all defined over a field $k_0 \subset k$, then the birational map $X \to Y$ is also defined over k_0.

This follows from the proof, since by a linear transformation over k_0, the canonical images (and the branch set on them in the hyperelliptic case) can be superposed.[30]

The remark is useful, in the case $k_0 = \boldsymbol{R}$ is the real field. It gives a *Torelli theorem for* the existence of *conformal mapping* of a Riemann surface with boundaries into another with the same genus and the same number of connected boundaries.

10. Some remarks, not indispensable, for the case of characteristic $p \neq 2$. a) We want to show that the above proof essentially holds true for any characteristic $p \neq 2$.

Note first that, in the hyperelliptic case, nothing has to be changed. In the non-hyperelliptic case, we have to investigate to see if the set of tangent lines at the non-singular points of an irreducible algebraic plane curve are able to determine the curve uniquely. Since the question is of a local nature, we may suppose the given curve C to be the locus of a general point (x, y) in the affine plane. Let $f(x, y) = 0$ be the equation of C. The tangent line at (x, y), $(X - x)\partial f/\partial x + (Y - y)\partial f/\partial y = 0$, has the coordinates

[28] A rational normal curve C in P_n is uniquely determined by any $n+3$ distinct points of it. Cf. Bertini, Complementi di Geometria Proiettiva, Zanichelli, Bologna (1927), p. 67.

[29] This again because $p \neq 2$.

[30] Notice that the genus of X, Y does not change by extending the field of constants from k_0 to k, hence the linear spaces of differentials over X, Y and on the symmetric product all have the same dimension as over k so that the canonical mappings are defined over k_0. (See the book of C. Chevalley, *Introduction to the Theory of Algebraic Functions of One Variable*, American Math. Soc. (1951); in particular, Theorem 4, p. 30 and Theorem 5, p. 99).

$(-\{x\partial f/\partial x + y\partial f/\partial y\}, \partial f/\partial x, \partial f/\partial y)$. Suppose for instance $\partial f/\partial y \not\equiv 0$; then the rational functions

$$-(y - x(\partial f/\partial x)/(\partial f/\partial y)) = U, \qquad -(\partial f/\partial x)/(\partial f/\partial y) = V$$

are the coordinates of a general point of a curve Γ (if C is not a line[31]) called the "dual curve of C" or the "envelope of C." We have a rational map $\phi: C \to \Gamma$ given by the above formulae, which maps a general point of C in a general point of Γ.

LEMMA. *Let P be the center on C of a linear branch with tangent $l_P = 0$. Suppose $v_P(l_P) = q \not\equiv 0 \pmod p$. Then to this branch, ϕ associates a branch on Γ of lines, with center at $l_P = 0$ and with contact point on $l_P = 0$ at the point P.*

Suppose P at the origin with tangent there in $y = 0$. The given branch has an expansion of the type $y = c_q x^q + c_{q+1}x^{q+1} + \cdots$ with $c_q \not\equiv 0$. The transformed branch has the equations:

$$U = (q-1)c_q x^q + qc_{q+1}x^{q+1} + \cdots$$

$$V = -\{qc_q x^{q-1} + (q+1)c_{q+1}x^q + \cdots\}.$$

The contact point has the coordinate $x_c = (U/V)_{x=0}$; hence the assertion is proved.

COROLLARY. *If for a point P of C, $v_P(l_P) = 2$, then for the general point Q of C, $v_Q(l_Q) = 2$ and, if the characteristic p is not 2, the map ϕ is birational.*

We can take as general point the point $Q = \{\substack{x=a \\ y=\sum c_r a^r}$ (in the field $k\{\{z\}\}$ of power series with finite negative powers). If $q = 2$, then one easily verifies $v_Q(l_Q) = 2$. Moreover, the above lemma shows that if for the general point P, $v_P(l_P) = q$ is not divisible by p, then the coordinates of P are rational functions of the coordinates of the general point of Γ.[32]

b) To complete our argument, we have only to show that the general projection C' of the canonical image of a non-hyperelliptic curve X is in the condition of the corollary.

Now if C is an algebraic curve in P_r and C' is its general plane projection,

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[31] In homogeneous coordinates, if $\partial f/\partial x_i = \rho c_i$ with $c_i \in k$ and $\rho \not\equiv 0$. From the Euler formula $\sum(\partial f/\partial x_i)x_i = 0$, we get for the general point x of C, $\sum c_i x_i = 0$.

[32] Hence for a general point P of C either $v_P(l_P)$ is divisible by the characteristic p or $v_P(l_P) = 2$.

the mapping $C \to C'$ is birational. Denote by P, P' corresponding general points on C, C' and by $l=0$, $l'=0$ the corresponding tangent hyperplane and tangent line. Then $v_P(l) = v_{P'}(l')$. All this is an obvious consequence of the theorem on primitive elements.

For values of the genus $g \geqq 4$, we have the following:

PROPOSITION 11. *Let C be the canonical image of a non-hyperelliptic curve of genus $g \geqq 4$. Let P be a generic point on C and let $l_P = 0$ be the general tangent hyperplane to C at P. Then $v_P(l_P) = 2$.*

Suppose $v_P(l_P) \geqq 3$. The specialty index of the divisor $3P$ is therefore $i(3P) = g-2$. The linear series $|3P|$ has dimension 1.[33] Let ϕ be a non-constant rational function with the pole $3P$: $(\phi) \succ -3P$. The field $k(C)$ is a separable extension of $k(\phi)$ with $[k(C) : k(\phi)] = 3$. (If the extension were inseparable C would be rational.) Take another point Q such that $v_Q(\phi - \phi(Q)) \neq 3$; this forbids only a finite number of positions for Q since $k(C)/k(\phi)$ is separable. If for Q we have $v_Q(l_Q) \geqq 3$, then we have another rational function ψ, non-constant, with $(\psi) \succ -3Q$. Again $k(C)$ is a separable extension of degree 3 of $k(\psi)$. The composition field of $k(\phi)$ and $k(\psi)$ is $k(C)$ itself, since $k(\phi) \neq k(\psi)$ in $k(C)$. We apply the following remark:

Let $K = k(X)$ be the field of rational functions on an algebraic curve X. Suppose that $K = k(\phi, \psi)$ with $[K : k(\phi)] = n$, $[K : k(\psi)] = m$ (both extensions being separable). Then the genus of $X \leqq n \cdot m - n - m + 1$.[34]

If we apply this to $X = C$ for $n = m = 3$, we get the proposition for the values $g > 4$.

For $g = 4$, C is a curve which belongs to a surface F of second order in P_3 (as it is shown by the theorem of R. R.). The groups of the series $|3P|$, since $i(3P) = g-2 = 2$, belong to lines necessarily contained in F. Hence we cannot have on C more than 2 series of order 3 and dimension 1.

This settles the case $g \geqq 4$ (for characteristic $p \neq 2$). For the case $g = 3$, we have to consider only the case of characteristic $p = 3$ when all tangents to the canonical plane quartic curve are all inflection lines. This can be done as in a paper of L. Gauthier.[35]

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[33] "Projective dimension," i. e. $l(3P) = 2$.

[34] J. P. Serre [7, (b)], Ch. IV, n. 8.

[35] L. Gauthier [3]. Any such quartic, by suitable choice of homogeneous coordinates x_0, x_1, x_2, can be brought into the form $x_1x_2^3 + x_2x_0^3 + x_0x_1^3 = 0$. On this curve, the verification becomes obvious.

11. Proof of the theorem for the non-hyperelliptic case. We give here the proof of Torelli's theorem, valid for any characteristic p for the case in which the curves we consider are not hyperelliptic.

LEMMA. *The rational map $\phi: C' \to \Gamma$ (of the general plane projection C' of the canonical curve C onto its envelope Γ) is purely inseparable.*

This is an obvious consequence of Proposition 8.

Suppose now we have two curves C', C'' with the same envelope Γ; we want to show that C' and C'' are birationally equivalent.

Suppose $[k(C'):k(\Gamma)] = p^{\alpha'}$, $[k(C''):k(\Gamma)] = p^{\alpha''}$ so that $k(C')^{p^{\alpha'}} = k(\Gamma)$, $k(C'')^{p^{\alpha''}} = k(\Gamma)$. Suppose $\alpha' \geqq \alpha''$; then $k(C'') = k(C')^{p^{\alpha'-p^{\alpha''}}}$. We have to prove that in fact $\alpha' = \alpha''$. To do this, denote by σ the automorphism of the universal domain $\mathfrak{Q}$ given by the exponentiation to the power $p^{\alpha'-\alpha''}$. We have $k(C')^{\sigma} = k(C'')$. If $f(x',y') = 0$ is the equation of C', $f^{\sigma}(x'',y'') = 0$ will be the equation of C''. If M' is a general point of C', $M'' = M'^{\sigma}$ is a general point of C''. Now $k(\Gamma)$ is defined as the field

$$k(y' - x'(\partial f/\partial x')/(\partial f/\partial y'),\ (\partial f/\partial x')/(\partial f/\partial y')),$$

the generators being computed at M'. Applying the automorphism σ, we get the field

$$k^{\sigma}(y'' - x''\{(\partial f/\partial x')/(\partial f/\partial y')\}^{\sigma},\ \{(\partial f/\partial x')/(\partial f/\partial y')\}^{\sigma}).$$

Now $k^{\sigma} = k$ since k is perfect; moreover, $(\partial f/\partial x')^{\sigma} = \partial f^{\sigma}/\partial x''$ $(\partial f/\partial y')^{\sigma} = \partial f^{\sigma}/\partial y''$ (these partial derivatives being computed at $M'' = M'^{\sigma}$). Hence σ carries the extension $k(C')/k(\Gamma)$ over the extension $k(C'')/k(\Gamma)$. Hence we must have $p^{\alpha'} = p^{\alpha''}$. q. e. d.

12. Proof of the theorem for the hyperelliptic case. a) We have already remarked that the proof given for characteristic $p = 0$ holds without changes for any characteristic $p \neq 2$.

For $p = 2$, there is no hope to deduce the theorem from the knowledge of the branch set on the canonical image. In this case, we have to deal with a quadratic extension $K(y)$ of the field $K = k(x)$ generated by an equation of type $y^2 - y = Q(x)$, where $Q(x) \in k(x)$ is determined mod. a substitution of type $Q(x) \to Q(x) + Q_0{}^2(x) - Q_0(x)$ with $Q_0(x) \in k(x)$. The rational function $Q(x)$ may be assumed in the form $Q(x) = A(x)/B(x)$ with A, B in $k[x]$ without common factors and such that every prime divisor of B is of odd multiplicity. The knowledge of the branch set gives the prime divisors of B which is utterly insufficient.

b) To prove the theorem in this case, we make the following remarks:

(α) For a hyperelliptic curve X the model $(X)^{(g-1)}$ is intrinsically defined with respect to biregular transformations in terms of the field $k((X)^{(g-1)})$.

In fact, since the canonical mapping is regular (Proposition 9, Corollary 2), $(X)^{(g-1)}$ is biregularly equivalent to the graph Φ of the mapping. Now Φ is non-singular, hence locally normal, therefore Φ coincides with its own normalized model Φ. Because of Corollary 1 to Proposition 9, we have proved the assertion.

(β) The canonical system on $(X)^{(g-1)}$ has a fixed part in the subvariety which is the locus of the point $(P_1)+(P_1{}^{\sigma})+(P_3)+\cdots+(P_{g-1})$, where $P_1, P_3, \cdots, P_{g-1}$ are general independent points on X and σ is the automorphism of X into itself given by the $g^1{}_2$. (In the notation above, for $p=2$, $\sigma: (x,y)\to(x,y+1)$). This follows from Proposition 6.

Consider now the canonical map $\phi: (X)^{(g-1)}\to P_{g-1}{}^{*}$. The points $y\in P_{g-1}{}^{*}$ such that $\phi^{-1}(y)$ has the minimum number of distinct points are given by the images of the osculator hyperplanes to the canonical image C of X. The curve C is rational normal, and the osculator hyperplane at any point $P\in C$ is uniquely determined and meets C only at P. The locus of the points y is therefore a rational curve Γ of $P_{g-1}{}^{*}$.

The set $\phi^{-1}(\Gamma)$ consists of the curves Δ_h, $0\leqq h\leqq[(g-1)/2]$, which are the images on $(X)^{(g-1)}$ of the curve X by the mappings $f_h: Q\to h(Q)+(g-1-h)Q$. Only Δ_0, i.e. the diagonal of $(X)^{(g-1)}$, does not lie on the common part of the canonical system. Notice also that $[k(X): k(\Delta_0)]=p^{\alpha}$ [36] for some $\alpha\geqq 0$. Suppose now that for the curve Y, we have $k((Y)^{(g-1)})=k((X)^{(g-1)})$. Then we have a biregular map of $(Y)^{(g-1)}$ onto $(X)^{(g-1)}$. Hence the diagonal Δ_0' of $(Y)^{(g-1)}$ is birationally equivalent with the diagonal Δ_0 of $(X)^{(g-1)}$. Moreover, the same argument given in n. 11 shows that $[k(Y): k(\Delta_0')]=[k(X): k(\Delta_0)]$.[37] This proves that $k(Y)\approx k(X)$.

q.e.d.

An analogous remark applies here as at the end of n. 9 (k_0 perfect).

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[36] Actually p^{α} = greatest power of p dividing $g-1$ (cf. J. P. Serre [7, (b)], Ch. III, n. 14).

[37] Or because both degrees equal the greatest power of p dividing $g-1$.

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THE SPECTRA OF MULTIPLIER TRANSFORMS ON l^p.*

By A. Devinatz and I. I. Hirschman, Jr.[1]

1. Introduction. In this paper, we consider spectral problems for multiplier transforms defined on the sequence spaces l^p, $1 \leqq p \leqq \infty$. We collect the basic and elementary facts about these operators in § 2. We show that to every such operator T on l^p there corresponds a unique (a.e.), bounded, measurable function $T^\wedge(\theta)$ defined on the interval $[0,1]$. The converse is not necessarily true, except for $p=2$.

In § 3, we characterize a wide class of functions $T^\wedge(\theta)$ for which the spectrum of the corresponding multiplier transform T is the essential range of the function. This means that the spectrum of any such operator does not change when it is considered as acting different l^p. However, the point spectrum may change as p changes. In § 4, we give some theorems which lead to a characterization of the point spectrum of certain classes of operators in terms of sets of multiplicity for l^p. We also give an example which shows the change of point spectrum as p changes. In so doing, we partially answer, in the negative, a question posed by P. Hartman [2]. We close the paper in § 5 with two theorems which are closely connected with the problem of spectral synthesis in l^∞.

2. Preliminaries. We shall designate by l^p, $1 \leqq p < \infty$, the Banach space of sequences of complex numbers such that

$$\| a \|_p = [\sum_{-\infty}^{\infty} | a(n) |^p]^{1/p} < \infty .$$

By l^∞, we shall designate the bounded sequences with

$$\| a \|_\infty = \sup_n | a(n) |,$$

by l_0^∞, the subspace of l^∞ whose elements vanish at infinity, by l_0, the sequences which vanish off a finite set of integers, and by l, the class of all sequences.

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For $a, b \in l$ we shall write

$$a * b(n) = \sum_{k=-\infty}^{\infty} a(n-k)b(k),$$

whenever this is defined for all n. Also, we set

$$(a, b) = \sum_{-\infty}^{\infty} a(n)b(-n),$$

whenever this is defined. For future reference, we note that if $a, c \in l_0$, $b \in l$, then

$$(a * b, c) = (a, b * c).$$

If p and q are conjugate indices,[2] then (a, b) is a bilinear functional on $l^p \times l^q$ which satisfies the relation

$$|(a, b)| \leqq \| a \|_p \| b \|_q.$$

For fixed b, (a, b) is a continuous linear functional on l^p. Moreover, if $a \in l^p$, $1 \leqq p \leqq \infty$, then

$$\| a \|_p = \text{l. u. b.}_{b \in l_0, \| b \|_q = 1} |(a, b)|. \tag{2.1}$$

A linear transform T with domain l_0 and range in l which satisfies

$$T(a * b) = T(a) * b$$

for every $a, b \in l_0$ will be called a multiplier transformation. Let e be the identity element of l; i.e. $e(0) = 1$, $e(n) = 0$ if $n \neq 0$. Then for any $a \in l_0$,

$$T(a) = T(e * a) = T(e) * a = t * a, \tag{2.2}$$

where $t = Te$. Let

$$\| T \|_p = \text{l. u. b.} \| Ta \|_p / \| a \|_p, \qquad (1 \leqq p \leqq \infty), \tag{2.3}$$

the least upper bound being extended over all $a \in l_0$ except $a = 0$. If $1 \leqq p < \infty$, then l_0 is dense in l^p while if $p = \infty$, then l_0 is dense in l_0^∞; thus if $\| T \|_p$ is finite, T has a unique extension as a bounded linear transformation of l^p to itself for $1 \leqq p < \infty$ and of l_0^∞ to itself if $p = \infty$. Let us denote by $\mathfrak{P}(T)$ the set of indices p for which $\| T \|_p$ is finite. We first assert that if $p_1, p_2 \in \mathfrak{P}(T)$ and if $1/r = (1-\omega)(1/p_1) + \omega(1/p_2)$, where $0 < \omega < 1$, then $r \in \mathfrak{P}(T)$ and $\| T \|_r \leqq \| T \|_{p_1}^{1-\omega} \| T \|_{p_2}^{\omega}$. This is an application of the Riesz-Thorin convexity theorem. We next assert that if $p \in \mathfrak{P}(T)$

[2] By this, we mean $1/p + 1/q = 1$. We shall always use p and q to designate conjugate indices.

then the conjugate index $q \in \mathfrak{I}(T)$ and $\| T \|_p = \| T \|_q$. To see this, we note that if $a, b \in l_0$, then

$$(Ta, b) = (a * t, b) = (a, t * b),$$

and thus

$$|(Ta, b)| \leqq \| a \|_q \| t * b \|_p \leqq \| a \|_q \| T \|_p \| b \|_p.$$

Referring to (2.1) and (2.3), we see that $\| T \|_q \leqq \| T \|_p$. Since the roles of p and q can be reversed, we have $\| T \|_p \leqq \| T \|_q$, and thus $\| T \|_p = \| T \|_q$ as desired.

These two remarks show that if $\mathfrak{I}(T)$ is not empty, then it is an interval (open or closed) with end points $1 \leqq \mu \leqq \nu \leqq \infty$ which satisfy $1/\mu + 1/\nu = 1$. In particular, if $\mathfrak{I}(T)$ is not empty, then $2 \in \mathfrak{I}(T)$. We also note that the convexity and conjugacy properties described above imply that if $p \in \mathfrak{I}(T)$, then $\| T \|_p \geqq \| T \|_2$.

If $p \in \mathfrak{I}(T)$, then, as we have remarked, $q \in \mathfrak{I}(T)$, where q is the index conjugate to p. Thus, $Te = t$ belongs to l^q. By (2.2) $Ta = t * a$ for any $a \in l_0$. Using the fact that $t \in l^q$, it is easily seen that this formula holds for $a \in l^p$ if $1 \leqq p < \infty$ and for $a \in l_0^\infty$ if $p = \infty$. If $p = \infty$, so that $t \in l^1$, then the right side of the formula $Ta = t * a$ is meaningful for $a \in l^\infty$ and in fact provides a natural extension of T from l_0^∞ to l^∞, which we continue to denote by T. T considered as a transformation from l^∞ to l^∞ clearly has the same norm as T considered as a transformation from l_0^∞ to l_0^∞ (this latter is of course just $\| T \|_\infty$).

If $\mathfrak{I}(T)$ is not empty, then, since $2 \in \mathfrak{I}(T)$, $Te \in l^2$ and we get

$$t(n) = Te(n) = \int_0^1 e^{-2\pi i n\theta} t^\wedge(\theta)\, d\theta.$$

Here,

$$t^\wedge(\theta) = \sum_{-\infty}^{\infty} t(n) e^{-2\pi i n\theta} \qquad (\mathfrak{M}_2),$$

the symbol $\mathfrak{M}_2$ indicating that $t^\wedge(\theta)$ is a limit in the mean of order 2 of the partial sums of the series on the right. It is well known and easily obtained by standard arguments that $\| t^\wedge \|_\infty = \| T \|_2$, where $\| t^\wedge \|_\infty =$ essential sup $| t^\wedge(\theta) |$. Therefore, an earlier remark gives

$$\| t^\wedge \|_\infty \leqq \| T \|_p \qquad p \in \mathfrak{I}(T). \tag{2.4}$$

3. The spectrum of multiplier transforms. Let T_1 and T_2 be multiplier transforms on l^p, $p \in \mathfrak{I}(T)$, and let $T = T_1 T_2$. It is easily seen that, with an evident choice of notations, $t^\wedge(\theta) = t_1^\wedge(\theta) t_2^\wedge(\theta)$. In particular, the

bounded function corresponding to T^n is $t^\wedge(\theta)^n$. Consequently, if $p \in \mathfrak{F}(T)$, then by (2.3) of § 2,

$$\| (t^\wedge)^n \|_\infty \leqq \| T^n \|_p.$$

Since $\| (t^\wedge)^n \|_\infty = \| t^\wedge \|^n$, we have immediately

$$\| t^\wedge \|_\infty \leqq \liminf_{n \to \infty} \| T^n \|_p^{1/n}. \tag{3.1}$$

A little later we shall give a class of multiplier transforms for which the inequality in (3.1) may be replaced by equality.

If T is a bounded operator on l^p, then by the spectrum, $S_p(T)$, of T, we shall mean the class of all complex λ such that $T - \lambda I$ does not have a bounded inverse on l^p.

By the essential range, $R(f)$, of a function $f(\theta)$, we shall mean the class of all λ such that the set $[\theta \mid |f(\theta) - \lambda| < \epsilon]$ has positive Lebesgue measure for every $\epsilon > 0$.

If T is a multiplier transform on l^2, then it is clear that T has a bounded inverse on l^2 if and only if $0 \notin R(t^\wedge)$. Consequently, $S_2(T) = R(t^\wedge)$.

THEOREM 3.1. *If $p \in \mathfrak{F}(T)$, then $S_2(T) \subseteq S_p(T)$.*

Proof. First we note that if T is a multiplier transform on l^p to itself and T^{-1} exists and is bounded, then T^{-1} is a multiplier transform on l^p. For, if $a, b \in l_0$, then

$$T^{-1}(a * b) = T^{-1}((TT^{-1}a) * b) = T^{-1}T((T^{-1}a) * b) = (T^{-1}a) * b.$$

Now, if $\lambda \notin S_p(T)$, then $T_\lambda = T - \lambda I$ has a bounded inverse on l^p, and therefore on l^q, and finally on l^2. But this means $\lambda \notin S_2(T)$. Theorem 3.1 is essentially a special case of a result due to Halberg [14].

THEOREM 3.2. *Let $t^\wedge(\theta)$ be the bounded measurable function corresponding to T. If[3] $t^\wedge(\theta+)$ and $t^\wedge(\theta-)$ exist for every $\theta \in [0,1]$, then $S_2(T) = S_p(T)$ for[4] $p \in \mathfrak{F}(T)^0$.*

Proof. We first assert that given $\epsilon > 0$, there exists a finite set $\{I_k\}_1^n$ of non-overlapping open intervals such that $I - \bigcup_{k=1}^n I_k$ $(I = [0,1])$ consists of a finite number of points, and such that

[3] It is of course sufficient for $t^\wedge(\theta)$ to have this property after being suitably redefined on a set of measure zero.

[4] $\mathfrak{F}(T)^0$ is the interior of $\mathfrak{F}(T)$.

$$\text{l. u. b.}_{\theta,\phi \in I_k} |\, t^\wedge(\theta) - t^\wedge(\phi)\,| < \epsilon, \qquad k = 1, \cdots, n.$$

To obtain such a set of intervals, let us associate with each point θ of I an open interval J_θ' whose right hand end point is θ such that if $\phi \in J_\theta'$, then $|\, t^\wedge(\phi) - t^\wedge(\theta-)\,| < \epsilon/2$, and an open interval J_θ'' whose left hand end point is θ such that if $\phi \in J_\theta''$, then $|\, t^\wedge(\phi) - t^\wedge(\theta+)\,| < \epsilon/2$. Finally, let $J_\theta = J_\theta' \cup J_\theta'' \cup \{\theta\}$. For each θ, J_θ is an open interval containing θ. Since I is compact, there exist values $\theta_1, \cdots, \theta_r$ such that $I = \bigcup_{i=1}^{r} (J_{\theta_i} \cap I)$. If we take for the I_k's the open intervals of the decomposition of I formed by the end points of $J_{\theta_1}' \cap I, \cdots, J_{\theta_r}' \cap I$, $J_{\theta_1}'' \cap I, \cdots, J_{\theta_r}'' \cap I$, then these have the desired properties.

Without any loss of generality, it will be enough to show that $0 \notin S_2(T)$ implies $0 \notin S_p(T)$. The result will then follow by Theorem 3.1. Let $m = \text{ess. inf.}\,|\, t^\wedge(\theta)\,|$. If $0 \notin S_2(T)$, then $m > 0$. Given $\epsilon > 0$, let $\{I_k\}_1^n$ be a composition as described above. We choose θ_k in I_k, $k = 1, \cdots, n$, such that $m \leqq |\, t^\wedge(\theta_k)\,| \leqq \| t^\wedge \|_\infty$.

If $p_k^\wedge(\theta)$ is the characteristic function of I_k, we have

$$1/t^\wedge(\theta) = \sum_{k=1}^{n} p_k^\wedge(\theta)/\{t^\wedge(\theta_k) + p_k^\wedge(\theta)[t^\wedge(\theta) - t^\wedge(\theta_k)]\}.$$

Further, if P_k is the operator corresponding to $p_k^\wedge(\theta)$, then since $p_k^\wedge(\theta)$ is of total variation 2, by a theorem of Stečkin [12] (see also [8]), we get $\| P_k \|_p \leqq A(p) < \infty$, where $A(p)$ is independent of k.

We may as well assume there exists an $r \in \mathfrak{F}(T)$ such that $r \neq 2$. For, otherwise the theorem is trivial. We have

$$\| P_k(T - t^\wedge(\theta_k) I \|_r \leqq 2A(r) \| T \|_r \tag{3.2}$$

and

$$\| P_k(T - t^\wedge(\theta_k) I) \|_2 = \| p_k^\wedge(\theta)\,[t^\wedge(\theta) - t^\wedge(\theta_k)] \|_\infty, \tag{3.3}$$

where I is the identity operator. If p is any number between r and 2, there exists an $0 < \omega < 1$ for which $1/p = (1-\omega)(1/r) + (1/2)\omega$. By the Riesz-Thorin convexity theorem, (3.2) and (3.3), we get

$$\| P_k(T - t^\wedge(\theta_k) I) \|_p \leqq [2A(r) \| T \|_r]^{1-\omega} \epsilon^\omega.$$

If we choose ϵ so small that

$$[2A(r) \| T \|_r]^{1-\omega} \epsilon^\omega < m,$$

we see that by expanding

$$\{t^\wedge(\theta_k) + p_k^\wedge(\theta)[t^\wedge(\theta) - t^\wedge(\theta_k)]\}^{-1}$$

into a geometric series, the operator corresponding to this function must be bounded on l^p. Therefore, the operator on l^p corresponding to $1/t^\wedge(\theta)$ must be bounded, and it is clearly the inverse of T.

COROLLARY 3.3. *Under the conditions of Theorem* 3.2,

$$\| t^\wedge(\theta) \|_\infty = \lim_{n\to\infty} \| T^n \|_p^{1/n}.$$

Proof. It is well known that $\| T \|_p{}^\wedge = \lim \| T^n \|_p^{1/n}$, where $\| T \|_p{}^\wedge = \sup [|\lambda| \mid \lambda \in S_p(T)]$ (cf. [7], p. 75). Since T is a normal operator on the Hilbert space l^2, we get $\| t^\wedge \|_\infty = \| T \|_2 = \| T \|_2{}^\wedge$. By Theorem 3.2, we get $\| T \|_2{}^\wedge = \| T \|_p{}^\wedge$. Therefore, $\| t^\wedge \|_\infty = \| T \|_p{}^\wedge$, which gives the result.

This is the promised sharpening of (3.2).

COROLLARY 3.4. *Under the conditions of Theorem* 3.2,

$$S_p(T) = R(t^\wedge).$$

Proof. This follows immediately from the equation

$$S_p(T) = S_2(T) = R(t^\wedge).$$

Corollary 3.4 generalizes results of P. Hartman [2] and G. L. Krabbe [5], [6].

4. The spectrum of multiplier transforms continued. In this section, we shall discuss theorems about multiplier transforms which will give information about their point spectra. We recall that a complex number λ is in the point spectrum of T acting on l^p if $T - \lambda I$ is not one to one; i.e., there exists a non-zero $a \in l^p$ such that $(T - \lambda I)a = 0$. The point spectrum of T is closely connected with the concept of sets of uniqueness and sets of multiplicity for l^p. There are many possibilities for defining a set of uniqueness for l^p. Here we mention only two.

We say that a set $\mathcal{E} \subseteq [0,1]$ is an ordinary set of uniqueness for l^p if $a \in l^p$ and

$$\sum_{-\infty}^{\infty} a(n) e^{2\pi i n\theta} = 0, \qquad \theta \notin \mathcal{E},$$

implies $a = 0$, where the sum is taken as an ordinary limit.

If $1 \leqq p \leqq 2$, we say that $\mathcal{E}$ is a mean set of uniqueness for l^p if $a \in l^p$ and

$$q - \text{l.i.m.} \sum_{-N}^{N} a(n) e^{2\pi i n\theta} = 0, \qquad \theta \notin \mathcal{E},$$

implies $a=0$, *where* q—l. i. m. *stands for the limit in the mean of* L^q, *the space of* q*-summable functions on* $[0,1]$.

An ordinary or mean set of multiplicity for l^p *is a set which is not an ordinary or mean set of uniqueness for* l^p, *respectively.*

When $1 \leqq p \leqq 2$ the point spectrum problem is elementary because of the Hausdorff-Young-Riesz theorem and we have the following result. The proof is left to the reader.

THEOREM 4.1. *If* $p \in \mathfrak{I}(T)$, $1 \leqq p \leqq 2$, *then* λ *is in the point spectrum of* T, *operating on* l^p, *if and only if the set* $[\theta \mid t\hat{}(\theta) = \lambda]$ *is a mean set of multiplicity for* l^p.

We should remark here that no example has yet been given of a mean set of uniqueness for l^p which is of positive measure $(1 < p < 2)$.

When $p > 2$, the situation is more complex. Let

$$\mathcal{E}_{\epsilon\lambda} = [\theta \mid |\, t\hat{}(\theta) - \lambda\,| < \epsilon]$$

and let

$$\mathcal{E}_\lambda = \bigcap_{\epsilon>0} \bar{\mathcal{E}}_{\epsilon\lambda}.$$

THEOREM 4.2. *Let*[3] $p \in \mathfrak{I}(T)^0$ *and let* $t\hat{}(\theta)$ *satisfy the assumptions of Theorem* 3.2. *If* $a \in l^p$ *and* $(T - \lambda I)a = 0$, *then*

$$\sum_{-\infty}^{\infty} a(n) e^{2\pi i n\theta} = 0, \qquad \theta \notin \mathcal{E}_\lambda.$$

Proof. Without any loss of generality, we may suppose $\lambda = 0$. Let

$$a_h(n) = \begin{cases} a(n) \sin 2\pi nh / 2\pi nh, & n \neq 0 \\ 1. & n = 0. \end{cases}$$

Further, let

$$A(\theta) = a(0)\theta + \sum_{-\infty}^{\infty}{}' (a(n)/2\pi i n) e^{2\pi i n\theta},$$

this series converging absolutely. Then

$$A_h(\theta) = [A(\theta + h) - A(\theta - h)]/2h = \sum_{-\infty}^{\infty} a_h(n) e^{2\pi i n\theta}, \qquad |\, h\,| > 0.$$

Let $I = [u, v]$ be any closed interval in the complement of $\mathcal{E}_0$. By the proof of Theorem 3.2, it is easy to see that

$$p\hat{}(\theta) = \begin{cases} 1/t\hat{}(\theta) & \theta \in I \\ 0 & \theta \notin I \end{cases}$$

gives rise to a bounded multiplier transform P on $\mathfrak{I}(T)$. Therefore, since Ta_h and Pe are in l^2, we have, by Parseval's equality,

$$(Ta_h, Pe) = \int_0^1 A_h(\theta)\, t\hat{}(\theta)\, p\hat{}(\theta)\, d\theta = \int_\mu^\nu A_h(\theta)\, d\theta.$$

Now, $a_h \to a$, and therefore $Ta_h \to Ta$ in l^p. Hence,

$$\lim_{h\to 0}(Ta_h, Pe) = \lim_{h\to 0} \int_\mu^\nu A_h(\theta)\, d\theta = A(\nu) - A(\mu) = 0.$$

Consequently, $A(\theta)$ is constant on each interval of the complement of $\mathcal{E}_0$ and this in turn implies that $\sum a(n) e^{2\pi i n\theta} = 0$ on each such interval (see [13], p. 270).

The converse of Theorem 4.2 can be proved, provided more assumptions are put on $t\hat{}(\theta)$. This will be shown in the next theorem. We first need a definition.

A function $f(\theta)$, $\mu \leqq \theta \leqq \nu$, is said to be of bounded β-variation,[5] $\beta > 0$, if

$$V_\beta[f] = \sup[\sum_{k=0}^{n} |f(\theta_{k+1}) - f(\theta_k)|^\beta]^{1/\beta} < \infty,$$

where the supremum is taken over all finite decompositions of the form $\mu = \theta_0 < \theta_1 < \cdots < \theta_{n+1} = \nu$.

The reader may easily verify that for functions of bounded β-variation, $f(\theta+)$ and $f(\theta-)$ exist. Therefore, these functions can have at most a countable number of singularities and they satisfy the conditions of Theorem 3.2.

THEOREM 4.3. *Let $t\hat{}(\theta)$ be of bounded β-variation for some $\beta > 0$. If $p \in \mathfrak{F}(T)$, $2 < p < \infty$, $a \in l^p$ and*

$$\sum_{-\infty}^{\infty} a(n) e^{2\pi i n\theta} = 0, \qquad \theta \notin \mathcal{E}_\lambda,$$

then

$$(T - \lambda I)^m a = 0,$$

for m an integer and $m \geqq \beta(p-2)/2p$.

Proof. Without any loss of generality, we may assume $\lambda = 0$. Since $\mathcal{E}_0$ is a closed set, its complement is the union of at most a countable set of open intervals, $\{I_k\}$. As in Theorem 4.2, let

$$A(\theta) = a(0)\theta + \sum_{-\infty}^{\infty} (a(n)/2\pi i n) e^{2\pi i n\theta},$$

and

$$A_h(\theta) = [A(\theta+h) - A(\theta-h)]/2h = \sum_{-\infty}^{\infty} a_h(n) e^{2\pi i n\theta}, \quad h > 0.$$

[5] For a further discussion of β-variation and multiplier transforms see [3].

where, as before, $a_h(n)=a(n)s_h(n)$, where $s_h(0)=1$ and $s_h(n)=\sin 2\pi nh/2\pi nh$ if $n\neq 0$. If $I_k=(\mu_k,\nu_k)$ and $2h<(\nu_k-\mu_k)$, then $A_h(\theta)=0$ for $\theta\in[\mu_k+h, \nu_k-h]$.

As usual, let e be the identity element of l^p and $e_n(m)=e(m+n)$. We have

$$
\begin{aligned}
(a_h, T^m e_n) &= \int_0^1 A_h(\theta)\, t\hat{}(\theta)^m e^{-2\pi i n\theta}\, d\theta \\
&= \sum_{k=1}^{N} \int_{I_k} A_h(\theta)\, t\hat{}(\theta)^m e^{-2\pi i n\theta}\, d\theta \\
&\quad + \sum_{k=N+1}^{\infty} \int_{I_k} A_h(\theta)\, t\hat{}(\theta)\, e^{-2\pi i n\theta}\, d\theta.
\end{aligned} \tag{4.1}
$$

Since $a_h\to a$ in l^p and T^m is bounded, it follows that

$$
\lim_{h\to 0}(a_h, T^m e_n) = (a, T^m e_n) = T^m a(n). \tag{4.2}
$$

Further, we claim that

$$
\lim_{h\to 0}\int_{I_k} A_h(\theta)\, t\hat{}(\theta)^m e^{-2\pi i n\theta}\, d\theta = 0
$$

for any k. For,

$$
\Big[\int_{I_k} |\, A_h(\theta)\, t\hat{}(\theta)^m\,|\, d\theta\Big]^2 \leqq \int_0^1 |\, A_h(\theta)\,|^2\, d\theta \int_{H_k} |\, t\hat{}(\theta)\,|^{2m}\, d\theta,
$$

where $H_k=(\mu_k, \mu_k+h]\cup[\nu_k-h, \nu_k)$ for $2h<|\,I_k\,|\,(|\,I_k\,|=\nu_k-\mu_k)$. Also,

$$
\int_0^1 |\, A_h(\theta)\,|^2\, d\theta = \sum_{-\infty}^{\infty} |\, a_h(n)\,|^2 \leqq \Big[\sum_{-\infty}^{\infty} |\, a(n)\,|^p\Big]^{2/p}\Big[\sum_{-\infty}^{\infty} |\, s_h(n)\,|^{2p/(p-2)}\Big]^{(p-2)/p},
$$

and since $p/(p-2)>1$, we get, by an easy estimation,

$$
\sum_{-\infty}^{\infty} |\, s_h(n)\,|^{2p/(p-2)} = O(1/h).
$$

Consequently,

$$
\int_0^1 |\, A_h(\theta)\,|^2\, d\theta = O(h^{(2-p)/p}).
$$

Since

$$
\int_{H_k} |\, t\hat{}(\theta)\,|^{2m}\, d\theta = O(h),
$$

we have

$$
\int_{I_k} A_h(\theta)\, t\hat{}(\theta)^m e^{-2\pi i n\theta}\, d\theta = O(h^{2/p}) \to 0 \text{ as } h\to 0.
$$

Therefore, the limit of the first sum in (4.1) is zero. If we can show that the second sum in this equation is small for suitable values of m, provided

N is sufficiently large independently of h, then this combined with equation (4.2) will complete the proof of the theorem.

We have

$$\sum_{N+1}^{\infty} \int_{I_k} A_h(\theta) t^\wedge(\theta)^m e^{-2\pi i n\theta}\, d\theta = \sum_{1}^{\infty} \int_{J_k} A_h(\theta) t^\wedge(\theta)^m e^{-2\pi i n\theta}\, d\theta,$$

where J_k is one of the intervals $(\mu_r, \mu_r + h]$ or $[\nu_r - h, \nu_r)$ in case $2h < |I_r|$, or is I_r in case $2h \geqq |I_r|$, $r > N$. Since the J_k are mutually disjoint, we get

$$\begin{aligned} [\sum_{1}^{\infty} \int_{J_k} |A_h(\theta) t^\wedge(\theta)^m|\, d\theta]^2 &= [\int_{U_N} |A_h(\theta) t^\wedge(\theta)^m|\, d\theta]^2 \\ &\leqq \int_0^1 |A_h(\theta)|^2\, d\theta \int_{U_N} |t^\wedge(\theta)|^{2m}\, d\theta \\ &\leqq O(h^{2/p-1})[\int_{U_N} |t^\wedge(\theta)|^{2m\alpha}]^{1/\alpha} |U_N|^{1/\beta}, \end{aligned} \tag{4.3}$$

where $U_N = \cup J_k$, $|U_N|$ is the Lebesgue measure of U_N, $\alpha > 1$ and $1/\alpha + 1/\beta = 1$. If $J_k = (\mu_r, \mu_r + h]$, then $t^\wedge(\mu_r -) = 0$; if $J_k = [\nu_r - h, \nu_r)$, then $t^\wedge(\nu_r +) = 0$; if $J_k = (\mu_r, \nu_r)$, then $t^\wedge(\mu_r -) = t^\wedge(\nu_r +) = 0$. We shall designate the appropriate end point of J_k by θ_k and we have $t^\wedge(\theta_k \pm) = 0$, with the correct sign chosen, depending on the end point. Therefore,

$$\begin{aligned} \int_{U_N} |t^\wedge(\theta)|^{2m\alpha}\, d\theta &= \sum_{1}^{\infty} \int_{J_k} |t^\wedge(\theta)|^{2m\alpha}\, d\theta \\ &= \sum_{1}^{\infty} \int_{J_k} |t^\wedge(\theta) - t^\wedge(\theta_k \pm)|^{2m\alpha}\, d\theta. \end{aligned} \tag{4.4}$$

Now, in J_k, the supremum of $t^\wedge(\theta)$ exists and is of the form $t^\wedge(\theta_k' +)$ or $t^\wedge(\theta_k' -)$, where $\theta_k' \in \bar{J}_k$, the closure of J_k. Therefore,

$$\begin{aligned} \sum_{1}^{\infty} \int_{J_k} |t^\wedge(\theta) - t^\wedge(\theta_k \pm)|^{2m\alpha}\, d\theta &\leqq h \sum_{1}^{\infty} |t^\wedge(\theta_k' \pm) - t^\wedge(\theta_k \pm)|^{2m\alpha} \\ &\leqq h V_{2m\alpha}[t^\wedge]^{2m\alpha}. \end{aligned} \tag{4.5}$$

If we choose $m \geqq \beta/2\alpha$, then $V_{2m\alpha}[t^\wedge]$ is, by hypothesis, finite. If we take $\alpha = p/(p-2)$, then from (4.3), (4.4) and (4.5) we get

$$[\sum_{N+1}^{\infty} \int_{I_k} A_h(\theta) t^\wedge(\theta) e^{-2\pi i n\theta}\, d\theta]^2 = O(|U_N|^{2/p}).$$

Since $|U_N| \to 0$ as $N \to \infty$, the left side goes to zero independently of h. This completes the proof of our theorem.

Remark. The restriction $p > 2$ in Theorem 4.3 is really no essential restriction. While Theorem 4.3 is true for $p \leqq 2$, Theorem 4.1 gives a

slightly stronger result. Notice for the case $\beta = 2p/(p-2)$, Theorem 4.3 is a direct converse of Theorem 4.2.

The method of proof we have used in this theorem is an extension of a method used by H. Pollard in [10], and C. Herz [4].

Notice that if $t^\wedge(\theta)$ satisfies the conditions of Theorem 3.2 and λ is in the point spectrum of T acting on l^p, $p \in \mathfrak{I}(T)^0$, then $\mathcal{E}_\lambda$ is an ordinary set of multiplicity for l^p. Conversely, if $t^\wedge(\theta)$ satisfies the condition of Theorem 4.3 and $\mathcal{E}_\lambda$ is an ordinary set of multiplicity for l^p, $p \in \mathfrak{I}(T)$ $(1 < p < \infty)$, then λ is in the point spectrum of T.

Remark. For continuous functions $t^\wedge(\theta)$, the set $\mathcal{E}_\lambda$ coincides with the set of θ where $t^\wedge(\theta) = \lambda$. The same is true for functions of bounded β-variation if they they are normalized by the condition that $t^\wedge(\theta) = \lambda$ if either $t^\wedge(\theta+) = \lambda$ or $t^\wedge(\theta-) = \lambda$. More generally, if $t^\wedge(\theta)$ is any function such that $t^\wedge(\theta+)$ and $t^\wedge(\theta-)$ exist, then for fixed λ, $t^\wedge(\theta)$ may be normalized so that $\mathcal{E}_\lambda$ is the set where $t^\wedge(\theta) = \lambda$.

Suppose now that $t^\wedge(\theta)$ is a function of bounded variation on $[0,1]$. In [2], P. Hartman poses the following question: Is λ an eigenvalue for T acting on l^p if and only if $[\theta \mid t^\wedge(\theta) = \lambda]$ is of positive Lebesgue measure? The following considerations show that the necessity part of this question must have a negative answer for $p > 2$.

In [11], R. Salem exhibites a perfect set $\mathcal{E}$ of Lebesgue measure zero and a continuous singular measure $d\mu$ with spectrum $\mathcal{E}$ so that

$$a(n) = \int_0^1 e^{-2\pi i n\theta}\, d\mu(\theta)$$

belongs to l^q for every $q > 2$. Clearly, it is possible to construct a continuous function $t^\wedge(\theta)$ of bounded variation which vanishes on $\mathcal{E}$, and is different from zero on the complement of $\mathcal{E}$. We have

$$Ta(n) = t * a(n) = \int_0^1 e^{-2\pi i n\theta}\, t^\wedge(\theta)\, d\mu(\theta) = 0,$$

which gives our result.

5. Two theorems connected with spectral synthesis in l^∞. Let $a \in l^\infty$ and

$$B(\theta) = a(0)\theta^2/2 + \sum_{-\infty}^{\infty}{}' \left(a(n)/-4\pi n^2\right) e^{2\pi i n\theta}.$$

If $t \in l^1$, then H. Pollard [10] has shown[6] that $t * a = 0$ only if $B(\theta)$ is

[6] Actually, Pollard is working on the group of real numbers. However, his methods also work here without essential modication.

linear on each interval contiguous to the set of zeros of $t^\wedge(\theta)$. It is not known whether the converse statement is true or false. Here we shall prove the converse statement for certain classes of sequences in l^1. We refer the reader to Pollard's paper for a complete discussion of the problem of spectral synthesis.

Let $f(\theta)$ be a function defined on $[0,1]$ and I any subinterval of this set with end points μ and ν. We shall write $f(I)=f(\nu)-f(\mu)$ and $|I|=\nu-\mu$. Let $\{I_k\}$ be a set of non-overlapping intervals each of which is contained in $[0,1]$. *We shall say that $f(\theta)$ is β-absolutely continuous if for any $\epsilon>0$ there exists a $\delta>0$ so that $\sum|I_k|<\delta$ implies $\sum|f(I_k)|^\beta<\epsilon$.*

THEOREM 5.1. *Let $t\in l^1$, $t^\wedge(\theta)$ its Fourier series, T the corresponding multiplier transform and $a\in l^\infty$. If $B(\theta)$ is linear on each open interval contiguous to the set of zeros of $t^\wedge(\theta)$ and $t^\wedge(\theta)$ is β-absolutely continuous, then*

$$T^m a=0,$$

where m is an integer and $m\geqq\beta/2$.

The proof of this theorem is similar to the proof of Theorem 4.3 and we leave the details to the reader.

Another theorem whose proof does not seem to follow from the considerations of the proof of Theorem 4.3 concerns those functions $t^\wedge(\theta)$ which belong to an integral Lipschitz class. Although the proof of our theorem is quite easy, it does not seem to have been recorded. We say that $f(\theta)\in \mathrm{Lip}(\alpha,p)$ if there exists a constant A such that for every h,

$$\left[\int_0^1|f(\theta+h)-f(\theta)|^p\,d\theta\right]^{1/p}<A|h|^\alpha.$$

We need the following theorem which is essentially due to S. Bernstein. A proof may be found in Zygmund [13], p. 135.

BERNSTEIN'S THEOREM. *If*

$$\int_0^1|t^\wedge(\theta+h)-t^\wedge(\theta)|^2\,d\theta<A|h|^{2\alpha},$$

$2\alpha>1$, then there exists a constant B, independent of A and α such that

$$\sum_{2^{s-1}<|k|\leqq 2^s}|t(k)|<AB\,2^{-n(1/2-\alpha)}, \tag{5.1}$$

and consequently,

$$\sum_{-\infty}^{\infty}|t(n)|<|t(0)|+AB\sum_{-\infty}^{\infty}{}'\,2^{-n(1/2-\alpha)}. \tag{5.2}$$

THEOREM 5.2. *Let* $t \in l^1$, $t^\wedge \in \mathrm{Lip}(\alpha, 2)$, $\alpha > \frac{1}{2}$, *and* $a \in l^\infty$. *If* $B(\theta)$ *is linear on each open interval contiguous to the set of zeros of* $t^\wedge(\theta)$, *then* $Ta = t * a = 0$.

Proof. Consider the operator defined on the complex z-plane by the conditions

$$C_\rho z = \begin{cases} \rho z/|z| & \text{if } |z| \geqq \rho \\ z & \text{if } |z| < \rho \end{cases} \qquad (\rho > 0).$$

The operator C_ρ has been called by Beurling [1] a circular contraction. If we set $t^\wedge_\rho = C_\rho t^\wedge$, it is quite obvious that

$$\int_0^1 |t^\wedge_\rho(\theta+h) - t^\wedge_\rho(\theta)|^2 \, d\theta \leqq \int_0^1 |t^\wedge(\theta+h) - t^\wedge(\theta)|^2 \, d\theta. \tag{5.3}$$

From (5.3), the hypothesis of the theorem and Bernstein's theorem it follows that t and t_ρ belong to l^1.

Set

$$\begin{aligned} B_h(\theta) &= [B(\theta+2h) - 2B(\theta) + B(\theta-2h)]/4h \\ &= \sum_{-\infty}^{\infty} b_h(n) e^{2\pi i n\theta}, \qquad |h| > 0, \end{aligned}$$

where $b_h(0) = a(0)$, $b_h(n) = a(n)(\sin 2\pi nh/2\pi nh)^2$ if $n \neq 0$. For fixed ρ, it follows from the continuity of $t^\wedge(\theta)$ that if h is sufficiently small, then $B_h(\theta) = 0$ everywhere that $[t^\wedge(\theta) - t^\wedge_\rho(\theta)] \neq 0$. Therefore, by the Parseval theorem, we get

$$(b_h, [T - T_\rho]e_n) = \int_0^1 B_h(\theta)[t^\wedge(\theta) - t^\wedge_\rho(\theta)] e^{-2\pi i n\theta} \, d\theta = 0,$$

where $e_n(m) = 0$ if $n + m \neq 0$, $e_n(-n) = 1$. Now, for every m,

$$|b_h(m)| \leqq |a(m)| \text{ and } b_h(m) \to a(m) \text{ as } h \to 0.$$

Therefore,

$$\begin{aligned} (a, [T - T_\rho]e_n) &= \lim_{h\to 0} \sum_{m=-\infty}^{\infty} b_h(m)[T - T_\rho]e_n(-m) \\ &= \lim_{h\to 0} (b_h, [T - T_\rho]e_n) = 0. \end{aligned}$$

This means that

$$Ta = T_\rho a \tag{5.4}$$

Clearly, for every m, $t_\rho(m) \to 0$ as $\rho \to 0$. Therefore, from (5.3), the hypothesis of Theorem 5.2 and formulas (5.1) and (5.2) of Bernstein's theorem, we get

$$\| T_\rho \|_1 = \sum_{-\infty}^{\infty} |t_\rho(m)| \to 0 \quad \text{as } \rho \to 0.$$

This, combined with (5.4), gives $Ta = 0$, which completes the proof of the theorem.

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THE INTEGRAL REPRESENTATIONS OF QUADRATIC FORMS OVER LOCAL FIELDS.*

By O. T. O'Meara.

Dedicated to Artin on his sixtieth birthday.

A quadratic form q which is obtained from a given form Q by a linear substitution of any number of new variables is called a representation of Q. As a special case we have the notion of equivalence of forms, the linear substitution then being a change of variables that can be inverted over the basic coefficient ring. One of the problems of interest in the arithmetic theory of forms is to determine all the representations of a given quadratic form. Using p-adic theory, Hasse [4] solved this problem over the rational numbers. An account of the rational and p-adic theory can be found in the book by Jones [6]. In both these cases the basic rings are fields; much less is known about integral representation, but the special problem of integral equivalence has been solved locally, and complete solutions can be found by consulting the works of Minkowski, Jones, Pall and O'Meara that are listed at the end of this paper. Conditions for the p-adic integral representation of one unimodular form by another are given by Jones at the odd primes [6]. It is our intention here to give necessary and sufficient conditions that will determine the integral representations of an arbitrary quadratic form over any local field in which 2 is either a unit or a prime.

Complete conditions for field representation are known for local fields, and we make this the basis of our discussion. Our criteria for the integral representation of one form by another will be in terms of the field representations of forms derived from certain canonical decompositions of the given forms.

I. Basic Concepts.

Summary. Our basic field F will be a local field [1] with ring of integers $\mathfrak{o}$ and prime ideal $\pi\mathfrak{o}$. We assume that $\operatorname{char} F \neq 2$, and that the residue class field $\mathfrak{o}/(\pi\mathfrak{o})$ is finite. We denote the multiplicative group of F by F^*,

* Received January 17, 1958.

and the group of units by $\mathfrak{u}$. Later on, we make special assumptions about the behaviour of 2 in F.

We shall consider finite dimensional vector spaces $U, V, W, \cdots$ over F. If V is provided with a scalar product $x \cdot y \in F$, we call it a quadratic space. For an $\alpha \in F^*$, we shall write $\alpha \circ V$ to mean the new quadratic space given by putting $x \circ y = \alpha(x \cdot y)$ on the old vector space V. We shall use $\bigoplus_1^p V_\lambda$ to denote the orthogonal (direct) sum of the quadratic spaces V_λ $(1 \leqq \lambda \leqq p)$. The radical of V will be written rad V. A non-degenerate space is one in which rad $V = 0$. The quadratic space V has a radical decomposition $V = U \oplus \text{rad } V$ in which U is non-degenerate. Any non-degenerate subspace U of a non-degenerate space V is a component of V, i.e. there is a non-degenerate subspace W such that $V = U \oplus W$. Moreover, W is then uniquely determined as $U\perp$, and $U\perp\perp = U$. An isotropic vector is a non-zero vector x for which $x^2 = 0$. For subsets S_1, S_2 of V, we write

$$S_1 \cdot S_2 = \{s_1 \cdot s_2 \mid s_1 \in S_1, s_2 \in S_2\},$$

$$S_1{}^2 = \{s_1{}^2 \mid s_1 \in S_1\}.$$

For further details on quadratic spaces, the reader is referred to *Geometric Algebra*, Ch. III.

By a lattice L in V we mean an $\mathfrak{o}$-module in the vector space V with the following property: the coordinates of the elements of L have bounded denominators in any basis for V. If this property holds for a single basis it must hold for all bases. The set 0 is a lattice in the space $V = 0$. Any $\mathfrak{o}$-module in L is again a lattice in V. The vector space spanned by L is the set FL. If L is the direct sum $\sum L_\lambda$, then FL is the direct sum $\sum FL_\lambda$. If L has a basis $\langle x \rangle$, then $\langle x \rangle$ is also a basis for FL. If FL has a basis $\langle y \rangle$, there is a basis $\langle z \rangle$ for $L = \sum \mathfrak{o} z_\lambda$ such that $Fy_1 + \cdots + Fy_i = Fz_1 + \cdots + Fz_i$ holds when $1 \leqq i \leqq \dim FL$. We can therefore speak of the dimension $\dim L$. An $\mathfrak{o}$-linear map of l into L can be extended to an F-linear map of Fl into FL. If the original map is an isomorphism, so is the extended one. In particular, an isomorphism preserves dimension. Given any non-zero x in FL, there is an $\alpha \in F$ such that $\alpha x \in L$, $\alpha\pi^{-1} x \notin L$. We put $\alpha L = \{\alpha x \mid x \in L\}$.

Now consider the lattice L as a subset of the quadratic space V. For an $\alpha \in F^*$ we shall write $\alpha \circ L$ to mean the same set L, but now as part of the quadratic space $\alpha \circ V$. We define L as the orthogonal sum $\bigoplus_1^p L_\lambda$ if (i) L is the direct sum of the L_λ, (ii) the L_λ are mutually orthogonal. Note that

$FL = \oplus FL_\lambda$ if $L = \oplus L_\lambda$. We say that a sublattice K is a component of L if there is a sublattice J such that $L = K \oplus J$. We define the radical as the lattice $\operatorname{rad} L = \{x \in L \mid x \cdot L = 0\}$. We then have

$$F \operatorname{rad} L = \operatorname{rad} FL, \qquad \operatorname{rad} L = L \cap \operatorname{rad} FL.$$

We call L degenerate if $\operatorname{rad} L \neq 0$. If $L = J \oplus K$, then L is non-degenerate if and only if J and K are. If J is non-degenerate, then K is unique and equal to $J^\perp$. For any L we have $L = K \oplus \operatorname{rad} L$.

From now until the end of this section we assume that V and L are non-zero so that they possess bases, the first over F the second over $\mathfrak{o}$. Associated with any basis $\langle x \rangle$ of the quadratic space V is the quadratic form

$$q(\alpha_1, \cdots, \alpha_n) = (\alpha_1 x_1 + \cdots + \alpha_n x_n)^2,$$

and varying $\langle x \rangle$ produces the entire fractional class of q. By $d(V)$ we mean the well-defined image of $\det(x_i \cdot x_j)$ in $0 \cup F^*/(F^*)^2$. We shall sometimes mean by $d(V)$ a particular representative of $d(V)$, for example $\det(x_i \cdot x_j)$. In the same way we can associate a quadratic form q with a basis of L, and varying $\langle x \rangle$ produces the entire unimodular class of q. Moreover, every unimodular class can be obtained from some lattice in such a way. We call $(x_i \cdot x_j)$ the matrix associated with $\langle x \rangle$ and write $L \cong (x_i \cdot x_j)$. By $d(L)$ we mean the well-defined image of $\det(x_i \cdot x_j)$ in $0 \cup F^*/\mathfrak{u}^2$. We shall sometimes denote by $d(L)$ a particular representative $\det(x_i \cdot x_j)$. Note that $d(FL) = d(FK)$ if $d(L) = d(K)$. We see that

$$d(\oplus V_\lambda) = \prod d(V_\lambda), \qquad d(\oplus L_\lambda) = \prod d(L_\lambda);$$

we also have $\operatorname{rad} V \neq 0$ if and only if $d(V) = 0$, and the same for L.

Representations. Let V, V' be two quadratic spaces. We say that V' represents V, and write it $V \to V'$, if there is an F-linear mapping ϕ of V into V' such that $(\phi x)^2 = x^2$ for all $x \in V$. In virtue of the equation $x \cdot y = \frac{1}{2}\{(x+y)^2 - x^2 - y^2\}$ we deduce that $\phi x \cdot \phi y = x \cdot y$. We call ϕ an isometry if it is one-one and onto, and we write $V \cong V'$. If $\phi_1 : (V_1 \to V_1')$ and $\phi_2 : (V_2 \to V_2')$ are representations, there is a unique extension of ϕ_1, ϕ_2 to a representation

$$\phi : (V_1 \oplus V_2 \to V_1' \oplus V_2').$$

If ϕ_1, ϕ_2 are isometries, so is ϕ. Consider radical decompositions

$$V = U \oplus \operatorname{rad} V, \qquad V' = U' \oplus \operatorname{rad} V',$$

and define $\psi : (U \to U')$ by

$$\phi x = \psi x + z, \qquad \psi x \in U', \; z \in \operatorname{rad} V'.$$

Then ψ is a representation. Since U is non-degenerate, ψ can have no kernel so it is actually an isometry of U into U'. We have therefore proved that

$$U \oplus \operatorname{rad} V \to U' \oplus \operatorname{rad} V' \iff U \to U'.$$

The same discussion holds for lattices, except for the fact that we use an $\mathfrak{o}$-linear map ϕ to define a representation of L into L'. All the previous results for spaces carry over to lattices, including the extension of a representation to an orthogonal sum, and

$$K \oplus \operatorname{rad} L \to K' \oplus \operatorname{rad} L' \iff K \to K'.$$

We also see that a representation $\phi : (L \to L')$ can be extended uniquely to a representation $FL \to FL'$; if ϕ is an isometry, so is its extension.

Now let V, V' be non-zero so that they have bases $\langle x \rangle$, $\langle x' \rangle$, with associated quadratic forms q, q'. Consider the representation $\phi : (V \to V')$. If we write $\phi x_i = \sum t_{\lambda i} x_\lambda'$, we see that

$$x_i \cdot x_j = \phi x_i \cdot \phi x_j = \sum \sum t_{\lambda i} (x_\lambda' \cdot x_\mu') t_{\mu j}$$

and hence the matrix $(x_i' \cdot x_j')$ represents the matrix $(x_i \cdot x_j)$. Thus q' represents q. Reversing the argument shows that $V \to V'$ if $q \to q'$. Hence the problem of representation of forms is equivalent to the problem of representation of quadratic spaces. In the very same way we see that the integral representation problem for forms is equivalent to the representation problem for lattices. As for ourselves, we prefer the lattice approach to representation theory because of the geometrical insight it provides. We are not going to refer to the quadratic forms again in this paper.

Now consider the radical decompositions

$$V = U \oplus \operatorname{rad} V, \qquad V' = U' \oplus \operatorname{rad} V'.$$

In practice, one has constructive methods for determining such decompositions. And $V \to V'$ if and only if $U \to U'$. The same applies to lattices. We shall therefore assume from now on that all *given lattices or spaces are without radical.* Under this assumption, every representation is an isometry into. Note that lattices or spaces that arise from the given ones, for example by inclusion, may be degenerate. We also assume that all *given* lattices are non-zero, although zero lattices will be allowed to turn up in discussions. We make no such assumption about our spaces which can be zero or non-zero.

PROPOSITION 1. *$U \to V$ if and only if a replica of U is a component of V.*

PROPOSITION 2. *If $V_1 \oplus U_1 \to V_2 \oplus U_2$ with $V_2 \to V_1$, then $U_1 \to U_2$.*

Proof. We have $V_2 \oplus V^* \oplus U_1 \oplus W^* \cong V_2 \oplus U_2$ by the last proposition. Hence $V^* \oplus U_1 \oplus W^* \cong U_2$ by Witt's theorem. Hence $U_1 \to U_2$. q.e.d.

DEFINITION. *If $U \to V$, define (V/U) so that $V \cong U \oplus (V/U)$. Otherwise, put $(V/U) = \emptyset$. Define $L/l = FL/Fl$.*

By Witt's theorem, V/U is well-defined. We also have

$$(1) \qquad (V \oplus W)/(U \oplus W) \cong V/U, \qquad (L \oplus J)/(K \oplus J) \cong L/K$$

It is easily seen that $\alpha \circ (FL) = F(\alpha \circ L)$, $\alpha \circ (V/U) \cong (\alpha \circ V)/(\alpha \circ U)$, $\alpha \circ (L/l) \cong (\alpha \circ L)/(\alpha \circ l)$.

Proposition 1 reduces the problem of representation of spaces to one of finding spaces with given Hasse invariants. The latter question has been solved over local fields and we shall make use of this fact in an indirect way, by expressing criteria for lattice representation in the form $U \to V$ where U, V are derived from the given lattices. The following proposition, whose proof is immediate, is a poor relative of Proposition 1.

PROPOSITION 3. *$l \to L$ if and only if L contains a replica of l.*

Maximal lattices. Let $\mathfrak{p}$ be a non-zero ideal in F and let L be any lattice in the quadratic space V. We use $N(L)$ for the norm ideal generated by $x^2, x \in L$. Call L integral if $N(L) \subseteq \mathfrak{o}$. We say that L is $\mathfrak{p}$-maximal in V if (i) $N(L) \subseteq \mathfrak{p}$, (ii) $N(K) \supset \mathfrak{p}$ whenever $K \supset L$. Let l be any lattice in V with $N(l) \subseteq \mathfrak{p}$. By taking a suitably scaled orthogonal basis for $(Fl)^\perp$, it is easy to find another lattice $l' \supseteq l$ with $N(l') \subseteq \mathfrak{p}$ and $\dim l' = \dim V$. In particular, every maximal lattice in a space has the dimension of the space itself. We now prove that every lattice l of norm $N(l) \subseteq \mathfrak{p}$ is contained in a $\mathfrak{p}$-maximal lattice. We can assume that $\dim l = \dim V$. By replacing V by $\alpha \circ V$ if necessary, we can take $\mathfrak{p} \subseteq 2\mathfrak{o}$. Take a strictly increasing sequence $l \subset l_1 \subset l_2 \subset \cdots$, in which all lattices have norm in $\mathfrak{p}$. Then $|d(l)| < |d(l_1)| < |d(l_2)| < \cdots$. Now $l_\lambda \cdot l_\lambda \subseteq \mathfrak{o}$ since $N(l_\lambda) \subseteq 2\mathfrak{o}$. Hence $d(l_\lambda) \subseteq \mathfrak{o}$. Hence the sequence terminates. Hence l is contained in a $\mathfrak{p}$-maximal lattices. Eichler [3] uses maximal lattices in a slightly different sense, but the following proposition follows immediately from his results.

PROPOSITION 4. *Two $\mathfrak{p}$-maximal lattices in the same quadratic space are isometric.*

Proof. We can suppose that $\mathfrak{p} = \mathfrak{o}$. First let the space V be anisotropic. By Theorem 9.4 of [3], we see that

$$\{x \in V \mid x^2 \in \mathfrak{o}\}$$

is a lattice, and it clearly contains every integral lattice in V. Thus there is only one $\mathfrak{o}$-maximal lattice and there is nothing to prove.

Now let V be isotropic. First we prove that any $\mathfrak{o}$-maximal L has norm $\mathfrak{o}$. Let x_0 be an isotropic vector in L such that $x_0 \notin \pi L$. Suppose, if possible, that $x_0 \cdot L \subseteq \frac{\pi}{2}\mathfrak{o}$. Then

$$J = \{(\alpha/\pi)x_0 + y \mid \alpha \in \mathfrak{o}, y \in L\}$$

is also a lattice, $N(J) \subseteq \mathfrak{o}$, and $J \supset L$, which denies the maximality of L. Hence $x_0 \cdot L = \frac{1}{2}\mathfrak{o}$. Let $y_0 \in L$ be such that $x_0 \cdot y_0 = \frac{1}{2}$. If y_0^2 is not a unit, then $(x_0 + y_0)^2$ is a unit. Hence $N(L) = \mathfrak{o}$. This proves the contention. If K is another $\mathfrak{o}$-maximal lattice in V, we have $N(L) = N(K) = \mathfrak{o}$. But L, K are maximal in the sense of Eichler and their norms are the same, hence $L \cong K$ by Theorem 9.6 of [3].

Jordan forms. We call l π^i-modular, or simply modular, if

$$x \in l, x \notin \pi l \Longrightarrow x \cdot l = \pi^i \mathfrak{o};$$

we call a modular l proper if $N(l) = l \cdot l$, we call it improper if $N(l) \subset l \cdot l$. Improper lattices do not occur when 2 is a unit in F; proper and improper lattices always occur when 2 is not a unit. If 2 is a prime, $N(l)$ is $l \cdot l$ or $2(l \cdot l)$. For a π^i-modular l, we have $l \cdot l = \pi^i \mathfrak{o}$ and also

$$x \in Fl, x \notin l \Longrightarrow x \cdot l \supset \pi^i \mathfrak{o}.$$

It is easily seen that $l_1 \oplus l_2$ is π^i-modular if and only if l_1 and l_2 are π^i-modular. Also that $\pi^\lambda \circ l$ is $\pi^{i+\lambda}$-modular if l is π^i-modular. And l is unimodular if and only if $l \cdot l \subseteq \mathfrak{o}$ with $d(l)$ a unit. Any l with $l \cdot l \subseteq \mathfrak{o}$ is called totally integral.

Every l has a decomposition $l = l_1 \oplus \cdots \oplus l_t$ in which the l_λ are modular with $l_{\lambda+1} \cdot l_{\lambda+1} \subset l_\lambda \cdot l_\lambda$. We call this a Jordan decomposition of l. If $l = \oplus l_\lambda'$ is another such decomposition, then (i) it too must have t modular components, (ii) $l_\lambda \cdot l_\lambda = l_\lambda' \cdot l_\lambda'$, (iii) l_λ and l_λ' are both proper or both improper, (iv) $\dim l_\lambda = \dim l_\lambda'$. We go on to say any two lattices l, l' have the same type if their Jordan forms satisfy these four properties. We have proved in [9] that if l contains a π^i-modular lattice J, where $\pi^i \mathfrak{o} = l \cdot l$, then J is a component of L.

DEFINITION. *Put* $\mathfrak{R}l = \{x \in Fl \mid x \cdot l \subseteq \mathfrak{o}\}$.

PROPOSITION 5. *If l has Jordan form $l = \oplus l_\lambda$, then $\mathfrak{R}l = \oplus(\pi^{-s(\lambda)} l_\lambda)$, where $\pi^{s(\lambda)}\mathfrak{o} = l_\lambda \cdot l_\lambda$. In particular, $\mathfrak{R}l$ is a lattice.*

As consequences, we have $\mathfrak{R}\,\mathfrak{R}l = l$, and $\mathfrak{R}(l \oplus l') = \mathfrak{R}l \oplus \mathfrak{R}l'$.

PROPOSITION 6. *If* $\dim l = \dim L$, *then* $l \to L$ *if and only if* $\mathfrak{R}L \to \mathfrak{R}l$.

Since it is possible to give constructive methods for finding a Jordan form, we can always assume that a lattice is given in Jordan form when necessary.

Notation. $\bigoplus_{\varphi} l_\lambda = 0$, $\bigoplus_{\varphi} V_\lambda = 0$.

DEFINITION. *Let* $l = \oplus l_\lambda$ *be a Jordan form. Define* $\mathfrak{l}_i = \oplus l_\mu$ *where* μ *extends over values for which* $l_\mu \cdot l_\mu \supseteq \pi^i\mathfrak{o}$. *Let* $\mathfrak{l}_i^{\perp} = \oplus l_\mu$ *where* μ *extends over values for which* $l_\mu \cdot l_\mu \subset \pi^i\mathfrak{o}$.

Clearly $\mathfrak{l}_i = 0$ and $\mathfrak{l}_i^{\perp} = l$ if $l_1 \cdot l_1 \subset \pi^i\mathfrak{o}$, while $\mathfrak{l}_i = l$ and $\mathfrak{l}_i^{\perp} = 0$ if $l_t \cdot l_t \supseteq \pi^i\mathfrak{o}$. And $l = \mathfrak{l}_i \oplus \mathfrak{l}_i^{\perp}$ for any i. In the case of a lattice $L = \oplus L_\lambda$, we use $\mathfrak{L}_i$, $\mathfrak{L}_i^{\perp}$ instead. All these quantities depend not only on the lattice but also on the Jordan form.

In this chapter we have been establishing our basic ideas and interpreting the classical problem geometrically. Many of the ideas have already appeared in print and fuller details with proofs and references can be found in [9].

II. When 2 is a Unit.

Throughout this chapter, we shall assume that 2 is a unit in the basic field F. We start with an investigation that will lead to necessary conditions for a representation $l \to L$. Then we prove these conditions sufficient.

DEFINITION. *Let* $\mathfrak{p}$ *be any ideal in* F. *We say that* $l \to L \bmod \mathfrak{p}$ *if there is an* $\mathfrak{o}$*-linear map* ϕ *of* l *into* L *such that*

$$(\phi x)^2 \equiv x^2 \bmod \mathfrak{p} \qquad \text{for all } x \in L.$$

It follows from the equation $x \cdot y = \frac{1}{2}\{(x+y)^2 - x^2 - y^2\}$ *that* $\phi x \cdot \phi y \equiv x \cdot y \bmod \mathfrak{p}$.

PROPOSITION 7. *Let* $l = l_1 \oplus \cdots \oplus l_t$ *be a Jordan form. If* $l \to L \bmod \mathfrak{p}$ *with* $\mathfrak{p} \subset l_t \cdot l_t$, *then* $l \to L$.

Proof. Every lattice over a field of this type has an orthogonal basis, hence we can write $l = \mathfrak{o}z \oplus k$ with $z^2\mathfrak{o} = l \cdot l$. Let $x \to \bar{x}$ denote a representation of l into $L \bmod \mathfrak{p}$, so that $x^2 \equiv \bar{x}^2 \bmod \mathfrak{p}$. If $\dim k = 0$ we are through by Hensel's lemma, so assume $\dim k > 0$ and proceed by induction to $\dim l$.

We know that $\bar{l}$ is spanned by $o\bar{z}$ and $\bar{k}$; define

$$K = \{\bar{x} - ((\bar{x}\cdot\bar{z})/\bar{z}^2)\bar{z} \mid x \in k\}.$$

Since $x\cdot z = 0$, we have $\bar{x}\cdot\bar{z} \in \mathfrak{p}$, hence $(\bar{x}\cdot\bar{z})/\bar{z}^2 \in \pi o$. It follows that K is a lattice, that $o\bar{z}$ and K span $\bar{l}$, that $o\bar{z}\cdot K = 0$, and hence that $\bar{l} = o\bar{z} \oplus K$. Now define ϕ on k by

$$\phi x = \bar{x} - ((\bar{x}\cdot\bar{z})/\bar{z}^2)\bar{z}.$$

We see that ϕ is o-linear. And $(\phi x)^2 \equiv x^2 \bmod \mathfrak{p}$. Hence $k \to K \bmod \mathfrak{p}$. By the induction, $k \to K$. But $oz \to o\bar{z}$ by Hensel's lemma. Hence $l \to \bar{l} \subseteq L$. q. e. d.

PROPOSITION 8. *Let $l = \oplus l_\lambda$, $L = \oplus L_\lambda$ be Jordan forms. If $l \to L$, then $\mathfrak{l}_i \to \mathfrak{L}_i$ for all i.*

Proof. We have $l = \mathfrak{l}_i \oplus \mathfrak{l}_i^\perp$, $L = \mathfrak{L}_i \oplus \mathfrak{L}_i^\perp$. Let $\phi : (l \to L)$ be a representation. Define a mapping ψ on $\mathfrak{l}_i$ in the following way: for each $x \in \mathfrak{l}_i$, write $\phi x = \psi x + z$ with $\psi x \in \mathfrak{L}_i$, $z \in \mathfrak{L}_i^\perp$. It is easily verified that ψ is o-linear. Since $z^2 \in \pi^{i+1}o$ we have

$$x^2 = (\phi x)^2 \equiv (\psi x)^2 \bmod \pi^{i+1}.$$

Hence $\mathfrak{l}_i \to \mathfrak{L}_i \bmod \pi^{i+1}$. Hence $\mathfrak{l}_i \to \mathfrak{L}_i$ by Proposition 7. q. e. d.

LEMMA. *Let l be π-modular, and let $L = L_1 \oplus L_2$ have L_1 unimodular and L_2 π-modular or 0. If $Fl \to FL$, then $l \oplus J_1 \oplus J_2 \to L$, where J_1 is unimodular or 0, J_2 is π-modular or 0, and $\dim(l \oplus J_1 \oplus J_2) = \dim L$.*

Proof. We can suppose that $l \subseteq FL$. Since $\mathfrak{R}l$ is π^{-1}-modular by Proposition 5, we can find a π^{-1}-maximal lattice $J \supseteq \mathfrak{R}l$. But $\mathfrak{R}l$ is then a component of J, hence $J = \mathfrak{R}l \oplus \mathfrak{R}J_2 \oplus \mathfrak{R}J_1$ with $\mathfrak{R}J_2$ π^{-1}-modular or 0, and $\mathfrak{R}J_1$ unimodular or 0. Now let $K \supseteq \mathfrak{R}L_1 \oplus \mathfrak{R}L_2$ be π^{-1}-maximal. Then $K \cong J$ by Proposition 4. Hence

$$\mathfrak{R}L_1 \oplus \mathfrak{R}L_2 \to K \to \mathfrak{R}l \oplus \mathfrak{R}J_2 \oplus \mathfrak{R}J_1,$$

so the result follows by Proposition 6. q. e. d.

THEOREM 1. *Let $l = \oplus l_\lambda$ and $L = \oplus L_\lambda$ be Jordan forms. Then $l \to L$ if and only if $F\mathfrak{l}_i \to F\mathfrak{L}_i$ for all i.*

Necessity. By Proposition 8 we have $\mathfrak{l}_i \to \mathfrak{L}_i$, hence $F\mathfrak{l}_i \to F\mathfrak{L}_i$ for all i.

Sufficiency. The proof is by induction to $\dim L$. Taking large enough i shows that $\dim l \leqq \dim L$. Taking an i with $l_1 \cdot l_1 = \pi^i o$ shows that $l_1 \cdot l_1$

$\subseteq L_1 \cdot L_1$. We suppose that $\dim l > 0$. First let $\dim L = 1$. Then $\dim l = 1$. Let l be π^i-modular. Then $Fl = F\mathfrak{l}_i \to F\mathfrak{L}_i$ so that $FL = F\mathfrak{L}_i$. Hence $l \to L$.

Now let $\dim L > 1$. First suppose that $l_1 \cdot l_1 \subseteq \pi^2(L_1 \cdot L_1)$. Define

$$L' = (\pi L_1) \oplus L_2 \oplus \cdots \oplus L_\lambda \oplus \cdots.$$

Put $\pi^j \mathfrak{o} = l_1 \cdot l_1$. Then for $i < j$, we have $F\mathfrak{l}_i = 0$ so that $F\mathfrak{l}_i \to F\mathfrak{L}_i'$; while for $i \geqq j$ we have $F\mathfrak{L}_i = F\mathfrak{L}_i'$ and again $F\mathfrak{l}_i \to F\mathfrak{l}_i'$. So l, L' satisfy the conditions of the theorem. Repeating this, we arrive at L'' with l, L'' still satisfying the conditions of the theorem, with $L'' \subseteq L$, and with $\pi^2(L'' \cdot L'') \subset l \cdot l$. In effect, this allows us to assume that $\pi(L_1 \cdot L_1) \subseteq l_1 \cdot l_1 \subseteq L_1 \cdot L_1$. By suitably scaling Fl and FL, we can assume that $L_1 \cdot L_1 = \mathfrak{o}$; then $l_1 \cdot l_1$ is $\mathfrak{o}$ or $\pi\mathfrak{o}$.

Case 1. $l_1 \cdot l_1 = \mathfrak{o}$. Then $l_1 = \mathfrak{l}_0$ and $L_1 = \mathfrak{L}_0$. So $Fl_1 \to FL_1$, and we have a replica of l_1 in FL_1. Let $J \supseteq l_1$ be $\mathfrak{o}$-maximal in FL_1. Then $J \cong L_1$ by Proposition 4 since L_1 is also $\mathfrak{o}$-maximal. So there is a replica $\bar{l}_1$ of l_1 in L_1. Now $\bar{l}_1$ is a component of L_1 since $\bar{l}_1$ is unimodular, hence $L_1 = \bar{l}_1 \oplus J_1$ with J_1 unimodular or 0. Define $L' = J_1 \oplus \mathfrak{L}_0{}^\perp$ and $l' = \mathfrak{l}_0{}^\perp$. For $i \leqq 0$, we have $Fl_i' = 0 \to F\mathfrak{L}_i'$. For $i > 0$, we have

$$Fl_1 \oplus Fl_i' = Fl_i \to F\mathfrak{L}_i = F\bar{l}_1 \oplus F\mathfrak{L}_i',$$

and so $Fl_i' \to F\mathfrak{L}_i'$ by Proposition 2. Hence $l' \to L'$ by the induction, hence $l \to L$.

Case 2. $l_1 \cdot l_1 = \pi\mathfrak{o}$. Then $l_1 = \mathfrak{l}_1$. By the lemma we have $\bar{l}_1 \oplus J_1 \oplus J_2 \subseteq \mathfrak{L}_1$ where $\bar{l}_1 \cong l_1$, J_1 is unimodular or 0, and J_2 is π-modular or 0. Define $L' = J_1 \oplus J_2 \oplus \mathfrak{L}_1{}^\perp$ and $l' = \mathfrak{l}_1{}^\perp$. For $i \leqq 1$ we have $Fl_i' = 0 \to F\mathfrak{L}_i'$. For $i > 1$ we have

$$Fl_1 \oplus Fl_i' = Fl_i \to F\mathfrak{L}_i = F\bar{l}_1 \oplus F\mathfrak{L}_i'.$$

Hence $Fl_i' \to F\mathfrak{L}_i'$ for all i. Hence $l' \to L'$ by the induction, hence $l \to L$. q. e. d.

III. When 2 is a Prime.

We now assume that 2 is a prime in the basic field F, so that the residue class field $\mathfrak{o}/(2\mathfrak{o})$ is a finite field of characteristic 2. Since $\operatorname{char} F \neq 2$, it follows [1] that F is a finite extension of the 2-adic numbers. Since the residue class field is finite it must have a quadratic extension, hence there are units ϵ, δ such that $x^2 + \epsilon x + \delta$ is irreducible modulo 2, hence there is a unit $\rho = \delta\epsilon^{-2}$ such that $x^2 + x + \rho$ is irreducible in F, hence

$$1 - 4\rho \notin (F^*)^2. \tag{2}$$

Now let ρ' be another unit for which $1-4\rho' \notin (F^*)^2$. Then $x^2+x+\rho'$ is irreducible, so $x^2+x+\rho'$ is irreducible modulo 2 by Hensel's lemma [1]. Let α be a root of $x^2+x+\rho$ in the quadratic extension of $\mathfrak{o}/(2\mathfrak{o})$; then we have $a \in \mathfrak{o}/(2\mathfrak{o})$ and a non-zero $b \in \mathfrak{o}/(2\mathfrak{o})$ such that $a+\alpha b$ solves $x^2+x+\rho'$; thus

$$a^2+\alpha^2 b^2+a+\alpha b+\rho'=0;$$

but $\alpha^2+\alpha+\rho=0$, so that $a^2+a+b^2\rho+\rho'=0$ and $b^2=b$; hence $b=1$ and so

$$x^2+x+(\rho+\rho') \equiv 0 \bmod 2$$

has a solution a in $\mathfrak{o}$. Hence

$$(1-4\rho)(1-4\rho') \equiv 1-4(\rho+\rho') \equiv (1+2a)^2 \bmod 8.$$

By Hensel's lemma, we therefore have

$$(1-4\rho)/(1-4\rho') \in (F^*)^2. \tag{3}$$

To sum up, if ρ and ρ' are integers (and therefore units) for which $1-4\rho$ and $1-4\rho'$ are non-squares, then $1-4\rho'$ is a square times $1-4\rho$. Accordingly, we fix the quantity ρ from now on and we use it to describe all non-squares of the form $1+4\alpha$ with $\alpha \in \mathfrak{o}$.

Modular classification. In this section we explicitly determine all unimodular lattices. The 1-dimensional ones are easily found, with $L \cong (\epsilon)$ being typical. We therefore examine higher dimensions.

TABLE I

	L	$d(L)$	L^2	$(FL)^2$
$H(0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	-1	$2\mathfrak{o}$	F
$B(0)$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	-1	$\mathfrak{u} \cup 4\mathfrak{o}$	F
$H(\rho)$	$\begin{pmatrix} 2 & 1 \\ 1 & 2\rho \end{pmatrix}$	$-1+4\rho$	$2\mathfrak{u}\mathfrak{o}^2$	$2\mathfrak{u}F^2$
$B(\rho)$	$\begin{pmatrix} 1 & 1 \\ 1 & 4\rho \end{pmatrix}$	$-1+4\rho$	$\mathfrak{u}\mathfrak{o}^2$	$\mathfrak{u}F^2$
$\Xi(\epsilon,\delta)$	$\begin{pmatrix} \epsilon & 1 \\ 1 & 2\delta \end{pmatrix}$			

Binary case. We write down in a table a list of different binary lattices and give (without proof) the sets of numbers represented by them. We shall then prove that the table exhausts all binary unimodular lattices.
In the table, ϵ, δ are units. We shall not need L^2 and $(FL)^2$ in the last case. Note that two Ξ's may be isometric for different values of ϵ, δ.

PROPOSITION 9. *Every binary unimodular lattice is isometric to exactly one of the five lattices listed in Table I.*

Proof. By looking at the $d(L)$ and L^2 columns of Table I, we see that no two of the lattices can be isometric. Consider an arbitrary binary unimodular lattice K. Referring to the special basis given in (13) of [9], we see that

$$K \cong \begin{pmatrix} \epsilon & 1 \\ 1 & 2\alpha \end{pmatrix} \text{ or } K \cong \begin{pmatrix} 2\epsilon & 1 \\ 1 & 2\alpha \end{pmatrix}$$

with ϵ a unit and $\alpha \in \mathfrak{o}$. If $d(K) = -1$, then $FK \cong FB(0) \cong FH(0)$ so that $K \cong H(0)$ or $K \cong B(0)$ by Theorem 4.4 of [8]. If $N(K) = \mathfrak{o}$ with $d(K) = -1 + 4\rho$, it is easily verified that $1 \in K^2$ and so $K \cong B(\rho)$. If $N(K) = 2\mathfrak{o}$ with $d(K) = -1 + 4\rho$, it is easily verified that $2 \in K^2$ and so $K \cong H(\rho)$ by Theorem 14.3 of [8]. Finally, there is the possibility that $d(K) = -1 + 2\beta$ with β a unit; a determinantal consideration then shows that ϵ and α are units, hence $K \cong \Xi(\epsilon, \alpha)$. q.e.d.

Ternary case. Again referring to (13) of [9], we see that any ternary unimodular K has the form

$$K \cong H(0) \oplus (\epsilon) \text{ or } K \cong H(\rho) \oplus (\epsilon), \tag{4}$$

where ϵ is a unit. By Table I, the second of these is anisotropic; while the first is isotropic. Hence K can have exactly one of the decompositions appearing in (4). Once K is decomposed, the value of $\epsilon\mathfrak{u}^2$ is uniquely determined. Thus (4) gives a complete description of the ternary case.

Quaternary case. Here, as in the binary case, we start by giving a table of lattices and we then prove that every unimodular quaternary lattice appears exactly once. Again use ϵ, δ for units.

TABLE II

(i)	$H(0) \oplus B(0)$	$H(0) \oplus H(0)$	(ia)
(ii)	$H(0) \oplus B(\rho)$		
(iii)	$H(\rho) \oplus B(0)$	$H(\rho) \oplus H(0)$	(iiia)
(iv)	$H(\rho) \oplus B(\rho)$		
(v)	$H(0) \oplus \Xi(\epsilon, \delta)$		

PROPOSITION 10. *Every quaternary unimodular lattice is isometric to exactly one of the lattices in Table II. We have*

$$FH(0) \oplus FB(0) \cong FH(0) \oplus FH(0) \text{ and } FH(\rho) \oplus FB(0) \cong FH(\rho) \oplus FH(0).$$

With these exceptions, all lattices in the table generate non-isometric spaces.

Proof. The stated F-isometries are obvious. Let us first prove the last part. A determinantal argument rules out all space isometries of (i) except, possibly, with (iv); this is impossible since $FH(\rho) \oplus FB(\rho)$ is anisotropic by Table I. Similarly, (ii) reduces to proving that $FH(0) \oplus FB(\rho)$ and $FH(\rho) \oplus FB(0)$ cannot be isometric, and this follows from Witt's theorem and Table I. The remaining cases follow from determinantal arguments. So the last part is proved.

Now we prove there are no lattice isometries in the table. The previous discussion reduces this to proving that $H(0) \oplus B(0) \not\cong H(0) \oplus H(0)$ and $H(\rho) \oplus B(0) \not\cong H(\rho) \oplus H(0)$; both these relations follow by taking norms.

Finally, we prove that every quarternary unimodular L appears in the table. By (13) of [9], we can write $L = H(\alpha) \oplus J$ with $\alpha = 0$ or ρ and J binary; this establishes the result except when $\alpha = \rho$ and $J \cong \Xi(\epsilon, \delta)$ or $J \cong H(\rho)$, in which case we use (12) of [10] and the obvious isometry $H(\rho) \oplus H(\rho) \cong H(0) \oplus H(0)$. q. e. d.

Higher dimensions. Any such unimodular L has a decomposition $L = (\oplus H(0)) \oplus K$ in which K is ternary or quaternary, again by (13) of [9]. By Proposition 3 of [9], the lattice K is unique to an isometry.

Representation modulo $\mathfrak{p}$. Let $\mathfrak{p}$ be any o-ideal in F and let $\mathfrak{q}$ stand either for $\mathfrak{p}$ or for $\frac{1}{2}\mathfrak{p}$. We write

$$l \to L \bmod (\mathfrak{p}, \mathfrak{q}) \tag{5}$$

if there is an o-linear map ϕ of l into L such that

$$(\phi x)^2 \equiv x^2 \bmod \mathfrak{p}, \qquad \phi x \cdot \phi y \equiv x \cdot y \bmod \mathfrak{q}.$$

If $\mathfrak{q} = \mathfrak{p}$, we write $l \to L \bmod \mathfrak{p}$. If ϕ is an isomorphism onto, we write $l \cong L \bmod (\mathfrak{p}, \mathfrak{q})$ and $l \cong L \bmod \mathfrak{p}$ respectively. We note that if

$$\phi_1 : (l_1 \to L_1 \bmod \mathfrak{p}, \mathfrak{q}) \text{ and } \phi_2 : (l_2 \to L_2 \bmod \mathfrak{p}, \mathfrak{q}),$$

then there is a unique extension of ϕ_1, ϕ_2 to $\phi : (l_1 \oplus l_2 \to L_1 \oplus L_2 \bmod \mathfrak{p}, \mathfrak{q})$. If ϕ_1, ϕ_2 are one-one, then so is ϕ. If ϕ_1, ϕ_2 are onto, then so is ϕ.

PROPOSITION 11. *Let l and L be unimodular and suppose that $N(l) = N(L)$ and $\dim l = \dim L$. Then $l \cong L \bmod 2$.*

Proof. By Theorem 2.4 of [8], we can write l and L in canonical bases $\langle x\rangle$, $\langle y\rangle$ respectively. If $N(l) = \mathfrak{o}$ this means that both $\langle x\rangle$ and $\langle y\rangle$ are orthogonal bases, and we can assume that $x_\lambda^2 \equiv 1 \equiv y_\lambda^2 \bmod 2$. If $N(l) \subset \mathfrak{o}$, this means that $\mathfrak{o}x_{2\lambda-1} + \mathfrak{o}x_{2\lambda} \cong H(\alpha)$ so that we have $x_\lambda^2 \equiv 0 \equiv y_\lambda^2 \bmod 2$ and $x_\lambda \cdot x_\mu = y_\lambda \cdot y_\mu$. The $\mathfrak{o}$-linear map sending x_λ to y_λ is an isometry modulo 2. q. e. d.

LEMMA. *Let $l = l_1 \oplus l_2$ be totally integral with l_1 unimodular and either 1-dimensional or 2-dimensional improper. Suppose that $\phi: (l \to L \bmod \mathfrak{p}, \mathfrak{q})$ with $\mathfrak{q} \subset \mathfrak{o}$. Then ϕl_1 is unimodular and has the same type as l_1, and there is a decomposition $\phi l = \phi l_1 \oplus l_2^*$ with a representation $l_2 \to L \bmod(\mathfrak{p}, \mathfrak{q})$ of l_2 onto l_2^*.*

Proof. The proof that ϕl_1 has the same type as l_1 is left to the reader. Let us write $\phi x = \bar{x}$. First we suppose that $l_1 = \mathfrak{o}x$ so that $\bar{l}_1 = \mathfrak{o}\bar{x}$. Define

$$l_2^* = \{\bar{z} - ((\bar{x}\cdot\bar{z})/\bar{x}^2)\bar{x} \mid z \in l_2\}.$$

Then $l_2^* \subseteq \bar{l} \subseteq L$ is a lattice, $\bar{l}_1$ and $\bar{l}_2^*$ span $\bar{l}$, and $\bar{l}_1 \cdot l_2^* = 0$ so that $\bar{l}$ is the direct sum of $\bar{l}_1$ and l_2^*; hence $\bar{l} = \bar{l}_1 \oplus l_2^*$. The mapping

$$\psi z = \bar{z} - ((\bar{x}\cdot\bar{z})/\bar{x}^2)\bar{x}$$

is the required representation of l_2 onto $l_2^* \bmod (\mathfrak{p}, \mathfrak{q})$.

Now let l_1 be improper, so that $l_1 = \mathfrak{o}x + \mathfrak{o}y$ and $\bar{l}_1 = \mathfrak{o}\bar{x} + \mathfrak{o}\bar{y}$. Define

$$l_2^* = \{\bar{z} + \alpha\bar{x} + \beta\bar{y} \mid z \in l_2\},$$

where $\alpha \in \mathfrak{q}$, $\beta \in \mathfrak{q}$ are given by

$$\alpha = \frac{\bar{y}^2(\bar{x}\cdot\bar{z}) - (\bar{x}\cdot\bar{y})(\bar{y}\cdot\bar{z})}{(\bar{x}\cdot\bar{y})^2 - \bar{x}^2\bar{y}^2}. \qquad \beta = \frac{\bar{x}^2(\bar{y}\cdot\bar{z}) - (\bar{x}\cdot\bar{y})(\bar{x}\cdot\bar{z})}{(\bar{x}\cdot\bar{y})^2 - \bar{x}^2\bar{y}^2}.$$

Then $l_2^* \subseteq \bar{l} \subseteq L$ is a lattice, $\bar{l}_1$ and l_2^* span $\bar{l}$, and $\bar{l}_1 \cdot l_2^* = 0$ so that $\bar{l}$ is the direct sum of $\bar{l}_1$ and l_2^*; hence $\bar{l} = \bar{l}_1 \oplus l_2^*$. The mapping $\psi z = \bar{z} + \alpha\bar{x} + \beta\bar{y}$ is the required representation of l_2 onto $l_2^* \bmod (\mathfrak{p}, \mathfrak{q})$. This proves the lemma.

PROPOSITION 12. *Let l be unimodular and suppose that $l \to L \bmod (\mathfrak{p}, \mathfrak{q})$ under the mapping ϕ with $\mathfrak{p} \subseteq 8/N(l)$. Then $l \cong \phi l$.*

Proof. Using the fact that l has a canonical basis [8], we can find a decomposition $l = l_1 \oplus l_2$ of the type of the lemma such that $l_2 = 0$ or

$N(l_1) = N(l_2) = N(l)$. First let $l_2 = 0$. If l_1 is proper, the result follows from Hensel's lemma. If l_1 is improper, we observe that $l_1 \cong H(\alpha)$, $\bar{l}_1 \cong H(\beta)$, $d(l_1) = d(\bar{l}_1)$, so that $l_1 \cong \bar{l}_1$ by Table I. The general case follows by induction, using the lemma. q. e. d.

PROPOSITION 13. *Let* $l \to L \bmod (\mathfrak{p}, \mathfrak{q})$ *under the mapping* ϕ *with* l *unimodular and* $\mathfrak{q} \subset \mathfrak{o}$. *Then* $\dim \phi l = \dim l$, ϕl *is unimodular, and* $\phi l \cong l \bmod \mathfrak{p}$.

Proof. If we take $x \in l$, $x \notin 2l$, it is a consequence of the modularity of l that $\phi x \neq 0$, hence ϕ is one-one, so that $\dim l = \dim \bar{l}$. It is easily seen that l is unimodular and that $N(l) = L(\bar{l})$. First let l be improper. If $\mathfrak{p} = 2\mathfrak{o}$, we have $\mathfrak{q} = 2\mathfrak{o} = \mathfrak{p}$, hence $l \cong \bar{l} \bmod \mathfrak{p}$ by Proposition 11. If $\mathfrak{p} \subseteq 4\mathfrak{o}$, we have $l \cong \bar{l}$ by Proposition 12. Hence assume that l is proper. Write $l = l_1 \oplus l_2$ with l_1 1-dimensional. Apply the lemma and the result follows by induction. q. e. d.

PROPOSITION 14. *Let* l *and* L *be unimodular with* $l \cong L \bmod \mathfrak{p}$, *where* $\mathfrak{p} \subset \mathfrak{o}$. *Given any* x_0 *with* $x_0^2\mathfrak{o} = \mathfrak{p}$, *there is a* y_0 *with* $y_0^2\mathfrak{o} = \mathfrak{p}$ *such that* $l \oplus \mathfrak{o}x_0 = K \oplus \mathfrak{o}y_0$ *with* $K \cong L \bmod 2\mathfrak{p}$.

Proof. First let l be improper so that L is improper too. Both l and L have the same dimension since they are isomorphic. By Proposition 12, we can assume that $\mathfrak{p} = 2\mathfrak{o}$. By (13) of [9], we can write $l = \oplus H(\alpha)$, where the α's are 0 or ρ. Similarly, we have $L = \oplus H(\beta)$. Now it is easily seen that both $H(0)$ and $H(\rho)$ can be chosen as components of $H(\alpha) \oplus \mathfrak{o}z_0$ whenever $z_0^2\mathfrak{o} = 2\mathfrak{o}$. Hence $(\oplus H(\alpha)) \oplus \mathfrak{o}x_0$ has $\oplus H(\beta)$ as a component. This proves the improper case, so we now assume that l and L are proper. Write $l = l_1 \oplus l_2$ with l_1 1-dimensional, and $l_2 = 0$ or $N(l_2) = N(l_1) = N(l) = \mathfrak{o}$. By the lemma, $L = \bar{l}_1 \oplus l_2^*$ with $l_1 \cong \bar{l}_1 \bmod \mathfrak{p}$ and $l_2 \to l_2^* \bmod \mathfrak{p}$. By Proposition 13, this means that $l_2 \cong l_2^* \bmod \mathfrak{p}$. Now using the perfectness of the residue class field, we easily see that $l_1 \oplus \mathfrak{o}x_0 = K_1 \oplus \mathfrak{o}z_0$ with $K_1 \cong \bar{l}_1 \bmod 2\mathfrak{p}$. Using an inductive argument, we see that $l_2 \oplus \mathfrak{o}z_0 = K_2 \oplus \mathfrak{o}y_0$ with $K_2 \cong l_2^* \bmod 2\mathfrak{p}$. Hence $l \oplus \mathfrak{o}x_0 = K \oplus \mathfrak{o}y_0$ with $K \cong L \bmod 2\mathfrak{p}$. q. e. d.

PROPOSITION 15. *Let* $l = \bigoplus_1^t l_\lambda$ *be a Jordan form and suppose that* $l \to L \bmod (\mathfrak{p}, \mathfrak{q})$ *under* ϕ, *where* $\mathfrak{p} \subseteq \mathfrak{q} \subset l_t \cdot l_t$. *Then* ϕl *is non-degenerate and has the same type as* l *with a Jordan form* $\phi l = \bigoplus_1^t l_\lambda^*$ *in which* $l_\lambda^* \cong_\lambda \bmod \mathfrak{p}$.

Proof. By suitably scaling l and L, we can take l_1 unimodular. By Proposition 13, we can take $t > 1$. We proceed by induction to $\dim l$. Write

$l_1 = K_1 \oplus J_1$ with K_1 1-dimensional or improper 2-dimensional, and $J_1 = 0$ or $N(J_1) = N(K_1) = N(l_1)$; this is possible since l_1 has a canonical basis. This induces a decomposition $l = K_1 \oplus K_2$ in which K_2 has the Jordan form $K_2 = J_1 \oplus (\bigoplus_2^t l_\lambda)$ with J_1 possibly 0. We can then write $\phi l = \phi K_1 \oplus K_2^*$, and we have a representation $K_2 \to L \bmod (\mathfrak{p}, \mathfrak{q})$ of K_2 onto K_2^* by the lemma. Now $K_1 \cong \phi K_1 \bmod \mathfrak{p}$ with K_1 and ϕK_1 of the same type, by Proposition 13. By the induction, we have

$$K_2^* = J_1^* \oplus (\bigoplus_2^t l_\lambda^*)$$

with K_2^* non-degenerate and of the same type as K_2, $J_1^* \cong J_1 \bmod \mathfrak{p}$, and $l_\lambda^* \cong l_\lambda \bmod \mathfrak{p}$ for $\lambda \geqq 2$. Hence

$$\phi l = (\phi K_1 \oplus J_1^*) \oplus (\bigoplus_2^t l_\lambda^*)$$

with $\phi K_1 \oplus J_1^* \cong K_1 \oplus J_1 \cong l_1 \bmod \mathfrak{p}$. q. e. d.

Notation. Let $l = \oplus l_\lambda$, $L = \oplus L_\lambda$ be given Jordan forms. We have already defined the lattices $\mathfrak{I}_i$, $\mathfrak{I}_i^\perp$, $\mathfrak{L}_i$, $\mathfrak{L}_i^\perp$ in terms of these decompositions. We now put $\mathfrak{L}_{(i)} = \oplus L_\mu$, where the summation extends over all values of μ for which $N(L_\mu) \supseteq 2^i\mathfrak{o}$; and $\mathfrak{L}_{(i)}^\perp = \oplus L_\mu$ with $N(L_\mu) \subset 2^i\mathfrak{o}$. Thus we have $L = \mathfrak{L}_{(i)} \oplus \mathfrak{L}_{(i)}^\perp$ for any i. We could also define $\mathfrak{I}_{(i)}$, $\mathfrak{I}_{(i)}^\perp$ in the same way, but we will not need them for l. We define $\mathfrak{I}_{[i]} = \oplus l_\mu$, where μ extends over all values for which $l_\mu \cdot l_\mu \supseteq 2^i\mathfrak{o}$ and, in addition, that value of μ, if any, for which $l_\mu \cdot l_\mu = 2^{i+1}\mathfrak{o}$ with l_μ improper. We define $\mathfrak{I}_{[i]}^\perp = \oplus l_\mu$ with μ extending over all values for which $l_\mu \cdot l_\mu \subset 2^{i+1}\mathfrak{o}$ and, in addition, that value of μ, if any, for which $l_\mu \cdot l_\mu = 2^{i+1}\mathfrak{o}$ with l_μ proper. Thus we have $l = \mathfrak{I}_{[i]} \oplus \mathfrak{I}_{[i]}^\perp$. We see that

$$\mathfrak{L}_{(i)} = \mathfrak{L}_{i-1} \text{ or } \mathfrak{L}_i, \qquad \mathfrak{I}_{[i]} = \mathfrak{I}_i \text{ or } \mathfrak{I}_{i+1}. \tag{6}$$

We now define Δ_i for L in the following way. If L has a proper 2^{i+1}-modular component, put $\Delta_i = 2^{i+1}\mathfrak{o}$; failing this, put $\Delta_i = 2^{i+2}\mathfrak{o}$ if L has a proper 2^{i+2}-modular component; otherwise, put $\Delta_i = 0$. We define δ_i as the quantity Δ_i when computed for l instead of L. Put $D_i = d(\mathfrak{L}_i)\mathfrak{o}$ and $d_i = d(\mathfrak{I}_i)\mathfrak{o}$; if $\mathfrak{L}_i = 0$ put $D_i = 0$, and the same when $\mathfrak{I}_i = 0$. Using the invariance of the type of a lattice, we see that δ_i, Δ_i, d_i, D_i are independent of the Jordan forms that define them. We have

$$\Delta_i = 2^{i+1}\mathfrak{o} \text{ or } 2^{i+2}\mathfrak{o} \text{ or } 0.$$

If $\alpha \in F$, denote by $(\alpha) \oplus L$ the lattice $\mathfrak{o}x \oplus L$ with $x^2 = \alpha$; similarly, with a quadratic space U. For any ideal $\mathfrak{p} \subset F$, write $\mathfrak{p} \to U$ if there is an $\alpha \in U^2$ for which $\alpha\mathfrak{o} = \mathfrak{p}$. If $\mathfrak{p} = 0$, we always have $\mathfrak{p} \to U$; if $\mathfrak{p} \neq 0$, saying that $\mathfrak{p} \to U$ is the same as saying that U represents some number of ordinal $\operatorname{ord} \mathfrak{p}$. We shall write $\mathfrak{p} \to \beta$ if $\mathfrak{p} \to Fx$ with $x^2 = \beta$. It is of practical importance to be able to decide whether $\mathfrak{p} \to U$ or not, and we shall give a solution to this field problem at the end of the paper. In the meantime, we need the following result.

PROPOSITION 16. *If* $\dim U \geqq 3$, *then* $\mathfrak{p} \to U$.

Proof. Write $U = \oplus Fx_\lambda$ with $x_\lambda{}^2\mathfrak{o} = \mathfrak{o}$ or $x_\lambda{}^2\mathfrak{o} = 2\mathfrak{o}$; we can suppose that $\dim U = 3$. Then it clearly suffices to assume that $x_\lambda{}^2$ are either all units or all primes; and we can then scale things to make them all units. The result follows by applying (4) and Table I to the lattice $\oplus \mathfrak{o}x_\lambda$. q. e. d.

PROPOSITION 17. *If* $l \to L$, *there exists a* J *of dimension* $\dim L - \dim l$ *such that* $l \oplus J \to L$.

Proof. By Proposition 3, we can take $l \subseteq L$. Let $\langle x \rangle$ denote an orthogonal basis for $(Fl)^\perp$ such that $x_\lambda \in L$. Put $J = \oplus \mathfrak{o}x_\lambda$. q. e. d.

Lower type. One of the simplest invariants in the theory of isometry of lattices is the type of a lattice. This invariant does not characterize a quadratic lattice, but it lays a foundation on which to build other invariants. Motivated by this, we introduce a corresponding notion of "lower type" into the representation theory.

Definition. We say that l *has a lower type than* L *if the following hold for all* i:

$$\dim \mathfrak{l}_i \leqq \dim \mathfrak{L}_i, \tag{7}$$

$$d_i D_i \to 1 \quad \text{if } \dim \mathfrak{l}_i = \dim \mathfrak{L}_i, \tag{8}$$

$$\delta_i \subseteq \Delta_i + 2^{i+2}\mathfrak{o} \text{ and } \Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathfrak{o} \text{ if } \dim \mathfrak{l}_i = \dim \mathfrak{L}_i, \tag{9}$$

$$\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathfrak{o} \text{ if } \dim \mathfrak{L}_i - 1 = \dim \mathfrak{l}_i > 0 \text{ and } d_i D_i \to 2^{i+1}, \tag{10}$$

$$\delta_i \subseteq \Delta_i + 2^{i+2}\mathfrak{o} \text{ if } \dim \mathfrak{L}_i - 1 = \dim \mathfrak{l}_i > 0 \text{ and } d_i D_i \to 2^i. \tag{11}$$

It is clear that the above definition is independent of the Jordan forms chosen. The relation $\delta_i \subseteq \Delta_i + 2^{i+2}\mathfrak{o}$ says that L has a proper 2^{i+1}-modular component if l does; similarly, $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathfrak{o}$ says that l has a proper 2^i-modular component if L does. The conditions (7)-(11) are elementary

in the sense that one can determine by inspecting given Jordan forms when they are satisfied.

PROPOSITION 18. *Let $l \to L$. Suppose that $j \geqq i$ and that $\mathfrak{l}_i$ has the Jordan form $\oplus l_\mu \neq 0$. Then there exists a lattice $\oplus l_\mu^*$ with the same type as $\oplus l_\mu$ such that $\oplus l_\mu^* \subseteq \mathfrak{L}_j$ and $l_\mu^* \cong l_\mu \bmod N(\mathfrak{L}_j^\perp)$.*

Proof. We have $\mathfrak{l}_i \to l \to L$. Let ϕ denote the representation $\mathfrak{l}_i \to L$. Given any $x \in \mathfrak{l}_i$, write $\phi x = \psi x + z$ with $\psi x \in \mathfrak{L}_j$ and $z \in \mathfrak{L}_j^\perp$. It is easily verified that ψ is $\mathfrak{o}$-linear and provides a representation

$$\mathfrak{l}_i \to \mathfrak{L}_j \bmod \mathfrak{p}, \mathfrak{q},$$

where $\mathfrak{p} = N(\mathfrak{L}_j^\perp)$, $\mathfrak{q} \subset 2^j\mathfrak{o} \subseteq 2^i\mathfrak{o}$, and $\mathfrak{q}$ is $\mathfrak{p}$ or $\frac{1}{2}\mathfrak{p}$. Our result then follows from Proposition 15. q. e. d.

PROPOSITION 19. *If $l \to L$, then l has a lower type than L.*

Proof. Choose Jordan forms for L and l. Fix an integer i and put $\mathfrak{l}_i = \oplus l_\mu$. By Proposition 18, there is a lattice $\mathfrak{l}_i^* \subseteq \mathfrak{L}_i$ which has the same type as $\mathfrak{l}_i$. Hence $\dim \mathfrak{l}_i = \dim \mathfrak{l}_i^* \leqq \dim \mathfrak{L}_i$, and (7) is proved. To prove (8), we can assume that $d_i \neq 0$, hence that $\mathfrak{l}_i \neq 0$. Since $\mathfrak{l}_i$ and $\mathfrak{l}_i^*$ have the same type, we have $d_i = d_i^*$. But $d(\mathfrak{l}_i^*)d(\mathfrak{L}_i) \in (F^*)^2$ since $F\mathfrak{l}_i^* = F\mathfrak{L}_i$. Hence $d_iD_i \to 1$ so that (8) is proved.

To prove the second part of (9), we suppose, if possible, that $\Delta_{i-1} = 2^i\mathfrak{o}$ but $\delta_{i-1} \subseteq 2^{i+1}\mathfrak{o}$. Thus $\mathfrak{L}_i$ has a proper 2^i-modular component while $\mathfrak{l}_i$ does not. By Proposition 18, we can choose $\mathfrak{l}_i^* \subseteq \mathfrak{L}_i$ of the same type as $\mathfrak{l}_i$. Since $\dim \mathfrak{l}_i^* = \dim \mathfrak{L}_i$, we have $\mathfrak{R}\mathfrak{L}_i \subseteq \mathfrak{R}\mathfrak{l}_i^*$. But $N(\mathfrak{R}\mathfrak{L}_i) = 2^{-i}\mathfrak{o}$ while $N(\mathfrak{R}\mathfrak{l}_i^*) \subseteq 2^{-i+1}\mathfrak{o}$, which is impossible. Our supposition must therefore be rejected. Hence $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathfrak{o}$.

Now (10). Suppose, if possible, that $\Delta_{i-1} = 2^i\mathfrak{o}$ but $\delta_{i-1} \subseteq 2^{i+1}\mathfrak{o}$. By Proposition 18, there is a lattice $\mathfrak{l}_i^* \subseteq \mathfrak{L}_i$ with the same type as $\mathfrak{l}_i$. Let J denote a 1-dimensional lattice for which $F(\mathfrak{l}_i^* \oplus J) = F\mathfrak{L}_i$ and $\mathfrak{l}_i^* \oplus J \subseteq \mathfrak{L}_i$. Since $d_i = d_i^* \neq 0$, we have $d_i^*D_i \to 2^{i+1}$, and we can actually assume that $J = \mathfrak{o}x$ with $x^2\mathfrak{o} = 2^{i+1+2r}\mathfrak{o}$ for some $r > 0$. Then $\mathfrak{R}\mathfrak{L}_i \subseteq \mathfrak{R}J \oplus \mathfrak{R}\mathfrak{l}_i^*$. Now $N(\mathfrak{R}\mathfrak{l}_i^*) \subseteq 2^{-i+1}\mathfrak{o}$ while $N(\mathfrak{R}\mathfrak{L}_i) = 2^{-i}\mathfrak{o}$. This is impossible since $N(\mathfrak{R}J) = 2^{-i-1-2r}\mathfrak{o}$.

Finally, (11) and the first half of (9). We can assume that $l \subseteq L$; by taking a suitably scaled orthogonal basis for $(Fl)^\perp$, we can assume that $\dim l = \dim L$. Hence $\mathfrak{R}L \to \mathfrak{R}l$. We put $\mathfrak{R}L = L'$, $\mathfrak{R}l = l'$, and define $(\Delta_i) = \Delta_i$ if $\Delta_i = 2^{i+1}\mathfrak{o}$, $(\Delta_i) = 0$ otherwise. Similarly, define (δ_i). We see

that $\delta_i \subseteq \Delta_i + 2^{i+2}\mathfrak{o}$ and $(\delta_i) \subseteq (\Delta_i)$ are equivalent; also, $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathfrak{o}$ and $(\Delta_{i-1}) \subseteq (\delta_{i-1})$. We have

$$(\Delta_{-i-2}') = 2^{-2i-2}(\Delta_i), \qquad (\delta_{-i-2}') = 2^{-2i-2}(\delta_i),$$

and also,

$$\mathfrak{R}\mathfrak{L}_i^{\perp} = \mathfrak{L}_{-i-1}', \qquad \mathfrak{R}\mathfrak{I}_i^{\perp} = \mathfrak{I}_{-i-1}'.$$

First we do (11). This is trivial when $\dim \mathfrak{L}_i = \dim L$, so we assume that $\dim \mathfrak{L}_i < \dim L$. The given conditions imply that $\dim \mathfrak{I}_{-i-1}' - 1 = \dim \mathfrak{L}_{-i-1}' > 0$ and $D_{-i-1}'d_{-i-1}' \to 2^{(-i-1)+1}$; applying (10), we see that $(\delta_{-i-2}') \subseteq (\Delta_{-i-2}')$, hence $(\delta_i) \subseteq (\Delta_i)$. We prove the first half of (9) in the same way, using the second half of (9) instead of (10). q. e. d.

The necessity. In this section we prove a series of propositions that lead to necessary conditions for our main theorem on the representation $l \to L$.

PROPOSITION 20. *If $l \to L$, then $\mathfrak{I}_{[i]} \to \mathfrak{L}_{(i+2)}$ and $F\mathfrak{I}_{[i]} \to F\mathfrak{L}_{(i+2)}$.*

Proof. Write $\mathfrak{I}_{[i]} = \oplus l_\mu$. Using (6) and Proposition 18, we see that there is a lattice $\oplus l_\mu^* \subseteq \mathfrak{L}_{(i+2)}$ with the same type as $\mathfrak{I}_{[i]}$ and such that $l_\mu^* \cong l_\mu \bmod 2^{i+3}$. It follows from Proposition 12 that $l_\mu^* \cong l_\mu$. Hence $\mathfrak{I}_{[i]} \to \mathfrak{L}_{(i+2)}$, hence $F\mathfrak{I}_{[i]} \to F\mathfrak{L}_{(i+2)}$. q. e. d.

LEMMA. *If $l \to L$ and $\delta_i = 2^{i+2}\mathfrak{o}$, then $\mathfrak{I}_{[i]} \oplus (2^{i+2}\epsilon) \to \mathfrak{L}_{(i+2)}$ for some unit ϵ.*

Proof. Take $x_0 \in \mathfrak{I}_{[i]}^{\perp}$ such that $x_0^2\mathfrak{o} = 2^{i+2}\mathfrak{o}$. Then $K = \mathfrak{I}_{[i]} \oplus \mathfrak{o}x_0 \to l \to L$. Let this representation of K into L be denoted by ϕ. For any $x \in K$, put $\phi x = \bar{x} + w$ with $\bar{x} \in \mathfrak{L}_{(i+2)}$ and $w \in \mathfrak{L}_{(i+2)}^{\perp}$. The mapping $x \to \bar{x}$ of K into $\mathfrak{L}_{(i+2)}$ is $\mathfrak{o}$-linear, and

$$\bar{x}^2 \equiv x^2 \bmod 2^{i+3}, \qquad \bar{x} \cdot \bar{y} \equiv x \cdot y \bmod 2^{i+2}.$$

From this follows

$$\bar{x}_0^2\mathfrak{o} = 2^{i+2}\mathfrak{o}, \qquad \bar{x}_0 \cdot \bar{z} \in 2^{i+2}\mathfrak{o} \text{ if } z \in \mathfrak{I}_{[i]}.$$

We define

$$J = \{\bar{z} - ((\bar{x}_0 \cdot \bar{z})/\bar{x}_0^2)\bar{x}_0 \mid z \in \mathfrak{I}_{[i]}\}.$$

It is easily verified that J is a lattice, that J and $\mathfrak{o}\bar{x}_0$ span $\bar{K}$, and that $\bar{x}_0 \cdot J = 0$, so that $\bar{K}$ is the direct sum of J and $\mathfrak{o}\bar{x}_0$, hence $\bar{K} = J \oplus \mathfrak{o}\bar{x}_0$. The mapping $z \to \bar{z} - ((\bar{x}_0 \cdot \bar{z})/\bar{x}_0^2)\bar{x}_0$ provides a representation modulo 2^{i+2} of $\mathfrak{I}_{[i]}$ onto J.

Let us now write $\mathfrak{I}_{[i]} = \oplus l_\mu$. Then $J = \oplus l_\mu^*$ with $l_\mu^* \cong l_\mu \bmod 2^{i+2}$ by

Proposition 15. By Proposition 14, we can therefore write $\bar{K} = (\oplus l_\mu^{**}) \oplus \mathfrak{o}y_0$ with $l_\mu^{**} \cong l_\mu \bmod 2^{i+3}$ and $y_0{}^2\mathfrak{o} = 2^{i+2}\mathfrak{o}$. By Proposition 12, we have $l_\mu^{**} \cong l_\mu$. Hence $\mathfrak{I}_{[i]} \oplus \mathfrak{o}y_0 \to \mathfrak{L}_{(i+2)}$. q. e. d.

PROPOSITION 21. *If* $l \to L$, *then* $\mathfrak{d}_i \to \mathfrak{L}_{(i+2)}/\mathfrak{I}_{[i]}$.

Proof. If $\mathfrak{d}_i = 0$, the result follows from Proposition 20. If $\mathfrak{d}_i = 2^{i+2}\mathfrak{o}$, if follows from the lemma. So assume $\mathfrak{d}_i = 2^{i+1}\mathfrak{o}$. Thus l has a proper 2^{i+1}-modular component l_λ. If we write $\mathfrak{I}_{i+1} = \oplus l_\mu$, using (6) and Proposition 18, we find a lattice $\oplus l_\mu{}^* \subseteq \mathfrak{L}_{(i+2)}$ of the same type as $\mathfrak{I}_{i+1}$ and such that $l_\mu{}^* \cong l_\mu \bmod 2^{i+3}$. For $\mu < \lambda$, this means that $l_\mu{}^* \cong l_\mu$, hence $\mathfrak{I}_{[i]} \oplus l_\lambda{}^* \to \mathfrak{L}_{(i+2)}$, hence $\mathfrak{d}_i = 2^{i+1}\mathfrak{o} \to \mathfrak{L}_{(i+2)}/\mathfrak{I}_{[i]}$. q. e. d.

PROPOSITION 22. *If* $l \to L$, *then* $\mathfrak{D}_i \to \mathfrak{L}_{(i+2)}/\mathfrak{I}_{[i]}$.

Proof. By Proposition 20, we can assume that $2^m\mathfrak{o} = \mathfrak{D}_i \neq 0$, hence $2^m\mathfrak{o}$ is $2^{i+1}\mathfrak{o}$ or $2^{i+2}\mathfrak{o}$. Therefore $\mathfrak{L}_{(i+2)}$ must have a 2^m-modular proper component. Since $\mathfrak{I}_{[i]} \to \mathfrak{L}_{(i+2)}$, by Proposition 17, we have a lattice J for which $\dim J + \dim \mathfrak{I}_{[i]} = \dim \mathfrak{L}_{(i+2)}$ such that $\mathfrak{I}_{[i]} \oplus J \to \mathfrak{L}_{(i+2)}$. Since $\mathfrak{R}\mathfrak{L}_{(i+2)}$ has a proper 2^{-m}-modular component, we have

$$2^{-m}\epsilon \to \mathfrak{R}\mathfrak{L}_{(i+2)} \to \mathfrak{R}J \oplus \mathfrak{R}\mathfrak{I}_{[i]}$$

for some unit ϵ. Since $N(\mathfrak{R}\mathfrak{I}_{[i]}) \subseteq 2^{-i}\mathfrak{o} \subset 2^{-m}\mathfrak{o}$, we have $2^{-m}\mathfrak{o} \to F(\mathfrak{R}J)$, thus $\mathfrak{D}_i \to FJ \cong \mathfrak{L}_{(i+2)}/\mathfrak{I}_{[i]}$. q. e. d.

PROPOSITION 23. *If* $l \to L$ *with* $\mathfrak{L}_{(i+2)}/\mathfrak{I}_{[i]} \cong FH(0)$, *then* $\mathfrak{D}_i\mathfrak{d}_i \subseteq \mathfrak{d}_i{}^2$.

Proof. Suppose that $\mathfrak{d}_i = 2^{i+2}\mathfrak{o}$ and $\mathfrak{D}_i = 2^{i+1}\mathfrak{o}$. Then by the lemma, there is a unit ϵ such that $\mathfrak{I}_{[i]} \oplus (2^{i+2}\epsilon) \to \mathfrak{L}_{(i+2)}$. By Proposition 17, we have

$$\mathfrak{I}_{[i]} \oplus (2^{i+2}\epsilon) \oplus (-2^{i+2+2r}\epsilon) \to \mathfrak{L}_{(i+2)}$$

for some $r > 0$. Since $\mathfrak{L}_{(i+2)}$ has a 2^{i+1}-modular component, there is a unit δ for which

$$2^{-i-1}\delta \to (-2^{-i-2-2r}\epsilon) \oplus (2^{-i-2}\epsilon) \oplus \mathfrak{R}\mathfrak{I}_{[i]}.$$

It is easily verified that this is impossible. q. e. d.

PROPOSITION 24. *If* $l \to L$, *then* $2^i(1+4\omega) \to (2^i \oplus \mathfrak{L}_{(i+1)})/\mathfrak{I}_i$ *for* $\omega = \rho$ *or* 0.

Proof. If $\mathfrak{I}_i = 0$, the proof is trivial. Accordingly, let $\mathfrak{I}_i$ have the Jordan form $\oplus l_\mu \neq 0$. By Proposition 18, there is a lattice $\mathfrak{I}_i{}^* \subseteq \mathfrak{L}_{(i+1)}$ of the same type as $\mathfrak{I}_i$ and with Jordan form $\oplus l_\mu{}^*$ in which $l_\mu{}^* \cong l_\mu \bmod 2^{i+2}$. By Propo-

sition 14, we can find another decomposition $\oplus l_\mu^{**}$ of the same type such that

$$\oplus l_\mu^{*} \oplus (2^{i+2}) = \oplus l_\mu^{**} \oplus (2^{i+2}\epsilon),$$

where ϵ is a unit, $l_\mu^{**} \cong l_\mu \bmod 2^{i+3}$, and hence $l_\mu^{**} \cong l_\mu$. We therefore have

$$\mathfrak{l}_i \oplus (2^{i+2}\epsilon) \to \mathfrak{l}_i^{*} \oplus (2^{i+2}) \to \mathfrak{L}_{(i+1)} \oplus (2^{i+2}).$$

A determinantal argument shows that $\epsilon = 1$ or $1 + 4\rho$. Hence

$$F(\mathfrak{l}_i \oplus (2^i\epsilon)) \to F((2^i) \oplus \mathfrak{L}_{(i+1)}),$$

hence $2^i(1 + 4\omega) \to (2^i \oplus \mathfrak{L}_{(i+1)})/\mathfrak{l}_i$. q. e. d.

PROPOSITION 25. *If $l \to L$, then $2^i(1 + 4\omega) \to (2^i \oplus \mathfrak{L}_{(i+1)})/\mathfrak{l}_{[i]}$ for $\omega = \rho$ or 0.*

Proof. Along the same lines as the last proposition. q. e. d.

Representation by modular lattices. In this section, we determine all lattices represented by a given unimodular lattice L.

PROPOSITION 26. *Let l be a 2^i-modular lattice with $\dim l \leqq 2$. Then there exists a lattice J such that $l \subset J \subset Fl$ with J 2^{i-2}-modular when $\dim l = 1$ and 2^{i-1}-modular when $\dim l = 2$; J can be chosen proper when l is improper.*

Proof. The case $\dim l = 1$ is trivial. If $\dim l = 2$, we write $l = \mathfrak{o}x + \mathfrak{o}y$ with $x^2 = 2^i\alpha$, $x \cdot y = 2^i$, $y^2 = 2^{i+1}\beta$, where α and β are integers; we can take β a unit and $\alpha \in 2\mathfrak{o}$ when l is improper; the existence of such a basis follows from Table I. Put $J = \mathfrak{o}x + \mathfrak{o}(\frac{1}{2}y)$. This satisfies the condition required of it. q. e. d.

PROPOSITION 27. *If l is totally integral, there is a lattice $J_1 \oplus J_2$ such that $l \subseteq J_1 \oplus J_2 \subset Fl$ with J_1 unimodular or 0 and J_2 either 0 or 1-dimensional 2-modular.*

Proof. By looking at a canonical basis of each modular component of a Jordan decomposition of l, we see that $l = \oplus l_\lambda$ with the l_λ modular and $\dim l_\lambda \leqq 2$. The result follows from successive application of Proposition 26. q. e. d.

THEOREM 2. *Let L be unimodular and let l be totally integral with $Fl \to FL$ and $N(l) \subseteq N(L)$. Then $l \to L$ if and only if*

$$(I) \qquad \Delta_{-1} \to L/\mathfrak{l}_{[-1]},$$

(II) $L/\mathfrak{l}_{[-1]} \cong FH(0)$ *implies* $\mathfrak{d}_{-1}\mathfrak{d}_{-1} \subseteq \mathfrak{d}_{-1}^2$.

Proof. The necessity is a consequence of Propositions 22 and 23. We prove the sufficiency. We can suppose that $l \subset Fl \subseteq FL$. By suitably scaling an orthogonal basis for $(Fl)^\perp$, we can find a lattice $J \subseteq (Fl)^\perp$ with $J \cdot J \subseteq 4\mathfrak{o}$ and such that $F(l \oplus J) = FL$. The values of $\mathfrak{l}_{[-1]}$ and $\mathfrak{d}_{-1}$ are the same for $l \oplus J$ as for l. In effect, this allows us to assume that $Fl = FL$ from the start. We make this assumption.

First we assume that L is improper. It is then easily verified that L is a 2-maximal lattice in FL. Since $N(l) \subseteq N(L)$, we can find a 2-maximal lattice J such that $l \subseteq J \subset FL$. But $J \cong L$ by Proposition 4. Hence $l \to J \to L$. Accordingly, we assume from now on that $N(L) = \mathfrak{o}$. We exclude the trivial case of $\dim L = 1$.

Let us suppose that l is unimodular too. If l is improper, we have

$$\mathfrak{o} = \mathfrak{d}_{-1} \to L/\mathfrak{l}_{[-1]} = L/l = 0,$$

which is impossible. Hence l and L have the same type, hence they are isometric by Theorem 4.4 of [8]. From now on we assume that $\mathfrak{l}_0 \neq l$.

Case 1. $N(l) = \mathfrak{o}$. Let $J_1 \oplus J_2 \supseteq l$ be a lattice as in Proposition 27. On determinantal grounds, we have $J_2 = 0$. Then $l \to J_1$; but J_1 is proper unimodular, so it is isometric to L by Theorem 4.4 of [8]. Hence $l \to L$.

Case 2. $\dim L$ odd. As in Case 1, we can find a unimodular lattice $J_1 \supseteq l$. But $N(J_1) = \mathfrak{o}$ since $\dim J_1$ is odd. Hence $l \to J_1 \to L$ by Theorem 4.4 of [8].

From now on, we can therefore assume that L is proper with $\dim L$ even, that $\mathfrak{l}_0 = \mathfrak{l}_{[-1]}$ is either 0 or improper unimodular, and that $\dim \mathfrak{l}_0^\perp \geqq 2$ is also even.

Case 3. $\dim \mathfrak{l}_0^\perp \geqq 4$. By Proposition 27, we can find $J \supseteq \mathfrak{l}_0^\perp$ with J a 2-modular lattice. By the decomposition (13) of [9], we can write $J = J_1 \oplus J_2$ with $J_1 \cong 2^\mathfrak{o} H(\alpha)$, where α is 0 or ρ. By Proposition 26, we can find a unimodular binary proper $J_1^* \supset J_1$. Hence, using Proposition 27 on J_2, we find a unimodular proper $J^* \supset J$. Then,

$$l \to \mathfrak{l}_0 \oplus J \to \mathfrak{l}_0 \oplus J^* \cong L.$$

Case 4. $\dim L = 2$. In this case, $\mathfrak{l}_0$ must be 0. First let L be anisotropic. By Proposition 27, we can find a unimodular binary lattice $J \supseteq l$. Then $l \to J$. A glance at Table I shows that L is either a $B(\rho)$ or one of the $\Xi(\epsilon, \delta)$, hence the same with J. Hence $l \to J \to L$ by Theorem 4.4 of [8].

Now let L be isotropic, hence we have $L \cong B(0)$. In view of the above reduction, we have $l \cdot l \subseteq 2\mathfrak{o}$. If we had $N(l) = 2\mathfrak{o}$, we would have $\Delta_{-1}\delta_{-1} \supset \delta_{-1}^2$ with $L/\mathfrak{I}_{[-1]} \cong FH(0)$, which is contrary to our hypothesis. Hence $N(l) \subseteq 4\mathfrak{o}$. Let J be a 4-maximal lattice such that $l \subseteq J \subset FL$. Then $2 \circ H(0) \to L$ is also 4-maximal and we have

$$l \to J \cong 2 \circ H(0) \to L.$$

This completes Case 4.

We are left with a proper unimodular L which has $\dim L \geqq 4$ even, and an l with an improper unimodular component $\mathfrak{I}_0$ with $\dim \mathfrak{I}_0^{\perp} = 2$.

Case 5. $\dim \mathfrak{I}_0^{\perp} = 2$. First let $\dim L > 4$. Using (13) of [9], we have $L \cong H(0) \oplus L'$ and $l \cong H(0) \oplus l'$. It is easily verified that l', L' satisfy the given conditions of the theorem. In effect, this allows us to assume that $\dim L = 4$. We then have $\mathfrak{I}_0 \cong H(0)$ or $H(\rho)$. First we consider an L which is neither an $H(0) \oplus B(0)$ nor an $H(\rho) \oplus B(0)$ in Table II. By Proposition 27, we can find a unimodular $J \supseteq l$. Then J itself has the form of one of the lattices appearing in Table II. Since $FJ = FL$, it follows from Proposition 10 and Theorem 4.4 of [8] that $J \cong L$, hence $l \to J \to L$.

If $L \cong H(0) \oplus B(0)$ and $\mathfrak{I}_0 \cong H(\rho)$, then $\mathfrak{o} = \Delta_{-1} \to L/\mathfrak{I}_{[-1]} \cong FH(\rho)$, which is impossible by Table I. Similarly, if $L \cong H(\rho) \oplus B(0)$, we cannot have $\mathfrak{I}_0 \cong H(0)$. Thus we consider $L \cong H(\alpha) \oplus B(0)$ and $\mathfrak{I}_0 \cong H(\alpha)$ with $\alpha = 0$ or ρ. Now it is easily verified that if we cancel these $H(\alpha)$ from L and l, then $B(0)$ and $\mathfrak{I}_0^{\perp}$ continue to satisfy the conditions of the theorem. But the theorem has already been established when $\dim L = 2$. Hence $\mathfrak{I}_0^{\perp} \to B(0)$, hence $l \to L$. q. e. d.

Method of proof. We can now give the necessity of our representation theorem over unramified 2-adic fields. This is no more than a restatement of Propositions 21-25. In virtue of Proposition 19, there will be no loss of generality in stating the theorem for a lattice l of lower type than L. The use of the symbols $\mathfrak{L}, \mathfrak{I}$ in the theorem presupposes that L, l are given in Jordan form. *Notation.* Write $\alpha(1 + 4\omega) \to U$ if either $\alpha \to U$ or $\alpha(1 + 4\rho) \to U$.

THEOREM 3. *Let l have a lower type than L. Then $l \to L$ if and only if the following conditions hold for all i:*

(I) $$\Delta_i \to \mathfrak{L}_{(i+2)}/\mathfrak{I}_{[i]},$$

(II) $$\delta_i \to \mathfrak{L}_{(i+2)}/\mathfrak{I}_{[i]},$$

(III) $$\mathfrak{L}_{(i+2)}/\mathfrak{I}_{[i]} \cong FH(0) \text{ implies } \Delta_i\delta_i \subseteq \delta_i^2,$$

(IV) $$2^i(1+4\omega) \to (2^i \oplus \mathfrak{L}_{(i+1)})/\mathfrak{l}_i,$$

(V) $$2^i(1+4\omega) \to (2^i \oplus \mathfrak{L}_{(i+1)})/\mathfrak{l}_{[i]}.$$

We are left with the task of proving the sufficiency. This will be done by an inductive argument on a quantity Φ to be introduced presently. Consider the lattice L given in Jordan form $L_1 \oplus \cdots \oplus L_T$. We put

$$\operatorname{len} L = \operatorname{ord} L_T \cdot L_T - \operatorname{ord} L_1 \cdot L_1.$$

For example, a modular lattice has length 0. We define

$$\Phi(L) = \dim L_1 + (\dim L)^2 + (\dim L)(\operatorname{len} L).$$

It is easily seen that

$$\min_L \Phi(L) = 2, \tag{12}$$

and this minimum it attained if and only if $\dim L = 1$. Now consider another lattice L' in Jordan form $\oplus L_\lambda'$ and suppose that

$$\dim L' \leqq \dim L, \operatorname{len} L' \leqq \operatorname{len} L, \dim L_1' \lessgtr \dim L_1. \tag{13}$$

Then it is easily verified that

$$\dim L_1' < \dim L_1 \Rightarrow \Phi(L') < \Phi(L), \tag{14}$$

$$\dim L' < \dim L \Rightarrow \Phi(L') < \Phi(L), \tag{15}$$

$$\operatorname{len} L' < \operatorname{len} L \Rightarrow \Phi(L') < \Phi(L). \tag{16}$$

Let us consider two lattices l, L which satisfy the conditions of the theorem. By either I or II, we have $F\mathfrak{l}_{[i]} \to F\mathfrak{L}_{(i+2)}$. If we choose i large enough, we have $\mathfrak{l}_{[i]} = l$, $\mathfrak{L}_{(i+2)} = L$, hence $Fl \to FL$. It follows from (7) that $l \cdot l \subseteq L \cdot L$. It follows from (9) that $N(l) \subseteq N(L)$. Hence

$$Fl \to FL, \qquad l \cdot l \subseteq L \cdot L, \qquad N(l) \subseteq N(L). \tag{17}$$

An elementary computation shows that $2^j \circ l$ has a lower type than $2^j \circ L$ and that

$$2^j \circ l, \qquad 2^j \circ L \text{ satisfy I-V.} \tag{18}$$

If L is modular, following (18), we can assume it is unimodular. Then $L = \mathfrak{L}_{(-1+2)}$ so that $l \to L$ by Theorem 2. Hence

$$l \to L \text{ if } L \text{ is modular.} \tag{19}$$

Now let l, L be the two lattices given in the theorem. We have to

prove $l \to L$. If $\Phi(L) = 2$ the result holds by (19). We proceed by induction. Thus we assume that $l' \to L'$ whenever l', L' satisfy the conditions of the theorem with $\Phi(L') < \Phi(L)$. We must deduce from this that $l \to L$. By (18) and (19), we can assume that

$$\text{(20)} \qquad \operatorname{len} L > 0, \qquad L \cdot L = L_1 \cdot L_1 = \mathfrak{o}.$$

In consequence, $l \cdot l = l_1 \cdot l_1 \subseteq \mathfrak{o}$. The assumption (20) is made from now on. Once we have proved our theorem, it will follow from the necessity that the given conditions continue to hold if we vary the Jordan forms of l and L. For the time being, we will have to content ourselves with the following two weaker forms of this result.

Remark 1. Let $\oplus L_\lambda'$ be another Jordan decomposition of L in which just two components L_1, L_2 have been changed, and suppose that $N(L_2) = 2\mathfrak{o}$ and that $\mathrm{I}_{[-1]} \to L_1'$. Then $\oplus L_\lambda'$ and $\oplus l_\lambda$ satisfy the conditions of the theorem.

Proof. Since the definition of lower type is independent of the Jordan from, we need only verify I-V. Of all quantities appearing in these conditions, the only ones to be affected by the new decomposition are $\mathfrak{L}_0$ and $\mathfrak{L}_{(0)}$. By inspection, I-III hold when $i = -2$, while for other values, there is nothing to prove since nothing is changed. Since $\mathrm{I}_{-1} = 0$, condition IV is trivial when $i = -1$. Condition V with $i = -1$ follows from the fact that $F\mathrm{I}_{[-1]}$ is a component of $F\mathfrak{L}_0' = FL_1'$. q. e. d.

Remark 2. Let $\oplus L_\lambda'$ be another Jordan form for L in which just two components L_1, L_k have been changed, and suppose that $N(L_k) = 4\mathfrak{o}$ and that $\mathrm{I}_{[-1]} = 0$. Then $\oplus L_\lambda'$ and $\oplus l_\lambda$ satisfy the conditions of the theorem.

Proof. As in the last remark, we need only verify I-V. Of all quantities appearing in these conditions, the only ones that can change are

$$\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_{(0)}, \mathfrak{L}_{(1)}.$$

Condition I must be examined at $i = -2, -1$, where it follows by inspection since $\mathrm{I}_{[-1]} = 0 = \mathrm{I}_{[-2]}$. If $i = -2$, δ_{-2} is either 0 or $\mathfrak{o}$ and II follows from (17). Similarly, for $i = -1$ when δ_{-1} is 0 or $\mathfrak{o}$. We consider $i = -1$ with $\delta_{-1} = 2\mathfrak{o}$; this is established at once using Proposition 16 and Table I, unless $\mathfrak{L}_{(1)}'$ is (ϵ') or $B(\rho)$, with ϵ' a unit. Now it is impossible to have $\mathfrak{L}_{(1)}' \cong (\epsilon')$, for then $\mathfrak{L}_{(1)} \cong (\epsilon)$ which contradicts the assumption $2\mathfrak{o} = \delta_{-1} \to \mathfrak{L}_{(1)}/\mathrm{I}_{[-1]} \cong \epsilon$. If $\mathfrak{L}_{(1)}' \cong B(\rho)$, using the fact that $N(L_k) \subseteq 4\mathfrak{o}$ and referring to Table I, we find that $\mathfrak{L}_{(1)} \cong B(\omega)$ with $\omega = 0$ or ρ. It is impossible to have $\omega = 0$

by III; it is impossible to have $\omega = \rho$ by II. This establishes II for the new decomposition.

To do III when $i = -2$, just observe that $\Delta_{-2}\delta_{-2} \subseteq \delta_{-2}^2$ always. Suppose we have $\mathfrak{L}_{(1)}'/\mathfrak{I}_{[-1]} \cong FH(0)$ with $\Delta_{-1} = \mathfrak{o}$ and $\delta_{-1} = 2\mathfrak{o}$. Then $\mathfrak{L}_{(1)}' \cong B(0)$. As above, this implies that $\mathfrak{L}_{(1)} \cong B(\omega)$ with $\omega = 0$ or ρ. As above, this is impossible. This proves III.

Condition IV holds for the new decomposition when $i = -1$ since $\mathfrak{I}_{-1} = 0$, so take $i = 0$. Since $L_1 \oplus L_k = L_1' \oplus L_k'$ with $N(L_k) = N(L_k') = 4\mathfrak{o}$, we see by Proposition 24 that $(1 + 4\omega') \oplus FL_1 \to (1) \oplus FL_1'$, hence

$$(1 + 4\omega') \oplus F\mathfrak{L}_{(1)} \to (1) \oplus F\mathfrak{L}_{(1)}',$$

$$(1 + 4\omega) \oplus F\mathfrak{I}_0 \to (1) \oplus F\mathfrak{L}_{(1)}.$$

Now add these representations, observe that

$$(1 + 4\omega) \oplus (1 + 4\omega') \cong 1 \oplus (1 + 4\omega''),$$

apply Proposition 2, and we find that

$$(1 + 4\omega'') \oplus F\mathfrak{I}_0 \to (1) \oplus F\mathfrak{L}_{(1)}',$$

so that IV remains true. It is seen in exactly the same way that V is true. q.e.d.

Remark 3. Let $\mathfrak{I}_0 = 0$. Suppose that $L_1 = J \oplus L_1'$ with J binary, and L_1' possibly 0. Let $K \subseteq J$ be a binary 2-modular lattice and put

$$L' = L_1' \oplus K \oplus \Big(\bigoplus_{\lambda \geqq 2} L_\lambda\Big).$$

Then $l \to L$ if L' is such that

(21) $N(K) = 4\mathfrak{o}$ when $\mathfrak{I}_{[0]} \neq 0$,

(22) $\delta_0 \subseteq \Delta_0' + 4\mathfrak{o}$ when L_1 is binary,

(23) $\delta_{-1} \to F\mathfrak{L}_{(1)}'$,

(24) $\mathfrak{L}_{(i+2)}'/\mathfrak{I}_{[i]} \cong FH(0)$ implies $\Delta_i'\delta_i \subseteq \delta_i^2$ for $i = -1, 0$.

Proof. We shall first verify that l, L' satisfy the conditions of the theorem, and then apply the inductive hypothesis to them. By inspection, we find

(25) $\Delta_i' = \Delta_i$ for $i \neq -2, -1, 0$,

(26) $F\mathfrak{L}_i' = F\mathfrak{L}_i$ for $i \neq 0$,

(27) $F\mathfrak{L}_{(i)}' = F\mathfrak{L}_{(i)}$ for $i \neq 0, 1$.

We also have

(28) $\Delta_0' = \Delta_0$ when $N(K) = 4\mathfrak{o}$,

(29) $F\mathfrak{L}_{(1)}' \supseteq F\mathfrak{L}_{(1)}$ when $N(K) = 2\mathfrak{o}$,

(30) $D_i' = \alpha_i^2 D_i$ with $\alpha_i \in F$.

Note that $D_i' \neq 0$ for $i \geqq 1$.

We start by proving that l has a lower type than L'. Then (7) follows from (26) since $\dim \mathrm{I}_0 = 0$, and (8) follows from (30) since $d_0 = 0$. The first half of (9) is immediate for $i \neq -2, -1, 0$, by (25) and (26); for $i = -2, -1$, it is trivial since $\mathrm{I}_0 = 0$; for $i = 0$, it is simply (22). The second half of (9) is deduced from (25) and (26) when $i \neq -1, 0, 1$; for $i = -1$, it is trivial; for $i = 0$, it holds since $\Delta_{-1}' \subseteq 2\mathfrak{o}$ when $\dim \mathfrak{L}_0' = \dim \mathrm{I}_0 = 0$; if $i = 1$ and $\mathrm{I}_{[0]} \neq 0$, it follows from (21) and (28); if $i = 1$ and $\mathrm{I}_{[0]} = 0$, then $\dim \mathrm{I}_1 = \dim \mathfrak{L}_1' > 0$ so that l_1 is 2-modular proper, hence $\delta_0 = 2\mathfrak{o}$, hence $\Delta_0 \subseteq \delta_0 + 4\mathfrak{o}$. Using (25) and (26), we get (10) for $i \geqq 2$; if $i = 1$ and $\mathrm{I}_{[0]} \neq 0$, this condition follows from (21); if $i = 1$ and $\mathrm{I}_{[0]} = 0$, the condition $\dim \mathrm{I}_1 > 0$ implies that l_1 is 2-modular and proper, hence $\Delta_0' \subseteq \delta_0 + 4\mathfrak{o}$. In proving (11), we can assume $i \geqq 1$ since $\dim \mathrm{I}_i > 0$; the condition then follows by (26), (30) and (25). This proves that l has lower type than L'.

Now we verify I-III. For $i \neq -2, -1, 0$, they follow from (25) and (27). Condition I follows trivially for $i = -2, -1$ since $\mathrm{I}_{[-2]} = \mathrm{I}_{[-1]} = 0$; similarly for $i = 0$ if $\mathrm{I}_{[0]} = 0$; if $\mathrm{I}_{[-0]} \neq 0$, then

$$\Delta_0' = \Delta_0 \to \mathfrak{L}_{(2)}/\mathrm{I}_{[0]} = \mathfrak{L}_{(2)}'/\mathrm{I}_{[0]}.$$

Condition II follows from I when $i = -2$, since $\delta_{-2} = 0$; for $i = -1$, II is simply (23); for $i = 0$, it follows from (27). Since $\delta_{-2} = 0$, III holds when $i = -2$; for $i = -1, 0$ it is (24).

By (27), we must check IV for $i = -1, 0$. It is then trivial since $\mathrm{I}_i = 0$. By (26), we must check V for $i = -1$; it is then trivial since $\mathrm{I}_{[-1]} = 0$.

All this proves that l and L' satisfy the conditions of the theorem. Clearly $\dim L' = \dim L$, $\mathrm{len}\, L' \leqq \mathrm{len}\, L$; we also have $\dim L_1' < \dim L_1$ when $\dim L_1 > 2$, and $\mathrm{len}\, L' < \mathrm{len}\, L$ when $\dim L_1 = 2$. By (13), (14) and (16) this means $\Phi(L') < \Phi(L)$. By the induction, we have $l \to L$. Hence $l \to L' \to L$. q.e.d.

We now continue with the proof. Exactly one of three different things can happen to l. Either $\mathrm{I}_{[0]} = 0$, or l_1 is unimodular, or l_1 is 2-modular improper. We find it convenient to consider these three possibilities separately in the rest of the proof.

When $\mathfrak{I}_{[0]} = 0$. In this section we find ourselves referring to Remark 3 a number of times. In doing so, it will be unnecessary to verify (21) since $\mathfrak{I}_{[0]} = 0$. The condition $\mathfrak{I}_0 = 0$ is also always true.

Case 1. L_1 proper, $\dim L_1 \geqq 3$. Then L_1 has an orthogonal basis; so by (4), we can write $L_1 = J \oplus L_1'$ with $J = \mathfrak{o}x + \mathfrak{o}y$, where $x^2 = 2$, $x \cdot y = 1$, $y^2 \in 2\mathfrak{o}$. In this case, $L_1' \neq 0$. Define $K = \mathfrak{o}x + \mathfrak{o}(2y)$. Define L' and apply Remark 3. Then (21), (22) are immediate. Since $\dim \mathfrak{L}_{(1)}' \geqq 3$, we have (23) in virtue of Proposition 16. Since $\dim \mathfrak{L}_{(i+2)}' \geqq 3$ and $\mathfrak{I}_{[i]} = 0$ for $i = -1, 0$, we have (24). Hence $l \to L$.

Case 2. $L_1 \cong \Xi(\epsilon, \delta)$. Write $L_1 = \mathfrak{o}x + \mathfrak{o}y = J$ with $y^2\mathfrak{o} = 2\mathfrak{o}$. Put $K = \mathfrak{o}(2x) + \mathfrak{o}y$. Define L' and apply Remark 3. We have $\Delta_0' = 2\mathfrak{o}$ so that (22) holds. Since $\delta_{-1} = 0$ or $2\mathfrak{o}$, and since L' has a proper 2-modular component, we have (23). A determinantal argument shows that $\mathfrak{L}_{(i+2)}'/\mathfrak{I}_{[i]}$ is not an $FH(0)$ for $i = -1, 0$, so that (24) holds. Hence $l \to L$.

Case 3. $L_1 \cong B(\alpha)$ with $\alpha = 0$ or ρ. First assume that $\Delta_0 = 2\mathfrak{o}$. Then we can write $L_1 \oplus L_2 = L_1' \oplus L_2'$ in such a way that $L_1 \cong \Xi(\epsilon, \delta)$ for some units ϵ, δ. By Remark 1, we see that $\oplus\, l_\lambda$ and $\oplus\, L_\lambda'$ satisfy the conditions of the theorem; hence $l \to L$ by Case 2. Accordingly, we now assume that $\Delta_0 \subseteq 4\mathfrak{o}$. Suppose that $\delta_0 = 2\mathfrak{o}$, so that $\delta_{-1} = 2\mathfrak{o}$ too. If $L_1 \cong B(\rho)$, we have $2\mathfrak{o} = \delta_{-1} \to \mathfrak{L}_{(1)}/\mathfrak{I}_{[-1]} \cong FB(\rho)$ which is impossible by Table I. If $L_1 \cong B(0)$, we have $\mathfrak{L}_{(1)}/\mathfrak{I}_{[-1]} \cong FH(0)$, hence $\Delta_{-1}\delta_{-1} \subseteq \delta_{-1}^2$, hence $\mathfrak{o} \cdot 2\mathfrak{o} \subseteq 4\mathfrak{o}$, which is false. Hence we deduce that $\delta_{-1} = 0$ and $\delta_0 \subseteq 4\mathfrak{o}$. We write $L_1 = J = \mathfrak{o}x + \mathfrak{o}y$ with $x^2 = 1$, $x \cdot y = 1$, $y^2 \in 4\mathfrak{o}$ and define $K = \mathfrak{o}(2x) + \mathfrak{o}y$. Define L' and apply Remark 3. Then (22) is immediate since $\delta_0 \subseteq 4\mathfrak{o}$. So is (23). So is (24) when $i = -1$; for $i = 0$, we have $\Delta_0' \subseteq 4\mathfrak{o}$, and therefore (24) holds. Hence $l \to L$.

Case 4. $L_1 \cong H(0)$, $\delta_0 = 4\mathfrak{o}$, $\Delta_0 \subseteq 4\mathfrak{o}$. We write $L_1 = \mathfrak{o}x + \mathfrak{o}y = J$ with $x^2 = 0 = y^2$, $x \cdot y = 1$. Put $K = \mathfrak{o}(2x) + \mathfrak{o}y$. Define L' and apply Remark 3. Then (21) and (22) are immediate. Since $\delta_{-1} = 0$, we have (23); for the same reason, we have (24) when $i = -1$; for $i = 0$, this condition follows from the fact that $\Delta_0' \subseteq 4\mathfrak{o}$. Hence $l \to L$.

Case 5. L_1 improper. By Case 4, we can assume that δ_0 is 0 or $2\mathfrak{o}$, or that Δ_0 is $2\mathfrak{o}$, when $L_1 \cong H(0)$. Write $L_1 = J \oplus L_1'$, $J = \mathfrak{o}x + \mathfrak{o}y$ with $x^2 = 2, x \cdot y = 1, y^2 \in 2\mathfrak{o}$. Put $K = \mathfrak{o}x + \mathfrak{o}(2y)$. Define L' and apply Remark 3. We see that $\Delta_0' = 2\mathfrak{o}$. Thus (21) and (22) are immediate. Since δ_{-1} is 0 or $2\mathfrak{o}$, we have (23). If $i = -1$, we have $\delta_{-1} = 0$ or $2\mathfrak{o}$ and $\Delta_{-1}' = 2\mathfrak{o}$,

so that $\Delta_{-1}'\delta_{-1} \subseteq \delta_{-1}^2$; if $i = 0$ with $\mathfrak{L}_{(2)}'/\mathfrak{I}_{[0]} \cong FH(0)$, then $F\mathfrak{L}_{(2)}' \cong FH(0)$ so that $L_1 \cong H(0)$ and $\Delta_0 \subseteq 4\mathfrak{o}$; according to our assumptions on δ_0, Δ_0, we therefore have $\delta_0 = 0$ or $2\mathfrak{o}$. Hence (24) holds and $l \to L$.

Case 6. $\dim L_1 = 1$, $\operatorname{len} L = 1$. In this case, we have $L_1 \cong (\epsilon)$, and L_2 2-modular. By (17), we can take $l \subseteq FL$. First we assume that L_2 is proper. Since $\mathfrak{I}_{[0]} = 0$, any improper component l_μ of l must have $l_\mu \cdot l_\mu \subseteq 4\mathfrak{o}$. Using the fact that each l_μ has a canonical basis, we can write $l = \oplus J_\lambda$ with $\dim J_\lambda \leqq 2$, $J_\lambda \cdot J_\lambda \subseteq 2\mathfrak{o}$, and $J_\lambda \cdot J_\lambda \subseteq 4\mathfrak{o}$ when J_λ is improper. Suitably scaling an orthogonal basis for $(Fl)^\perp$ allows us to assume that we have a lattice $J = \oplus J_\lambda$ with the above properties and such that $l \subseteq J$, $FL = FJ$. Using Proposition 26, we can find $J' = J_1' \oplus J_2'$ with J_1', J_2' both proper, J_1' 2-modular or 0, J_2' 4-modular, and $J' \supseteq J$. On determinantal grounds, J_2' cannot be 0; indeed, $\dim J_2'$ must be odd. If $\dim L = 2$, $\dim J_2' = 1$; using (4) and Proposition 26 allows us always to take $\dim J_2' = 1$. Now put $L_2' = 2L_1$, $L_1' = L_2$. Then it is easily seen (one may apply Theorem 9.1 of [8]) that $L' \cong J'$. Hence

$$l \to J \to J' \to L' \to L.$$

We must now consider the case of an improper L_2. We put $L' = (2L_1) \oplus L_2$. It is easily seen that L' is 4-maximal. If we had $N(l) = 2\mathfrak{o}$, we would have $2\mathfrak{o} \to \delta_{-1} \to \mathfrak{L}_{(1)}/\mathfrak{I}_{[-1]} \to FL_1$, which is impossible. Hence $N(l) \subseteq 4\mathfrak{o}$, so we can find a 4-maximal lattice J with $l \subseteq J \subset FL$. But $L' \cong J$ by Proposition 4. Hence $l \to J \to L' \to L$. This proves Case 6.

Case 7. $\dim L_1 = 1$, $\operatorname{len} L > 1$. Thus $L_1 \cong (\epsilon)$ with ϵ a unit. We put $L' = (2L_1) \oplus L_2 \oplus \cdots$. But for a simple regrouping of terms, this gives a Jordan form for L'. It is easily seen that

$$\Delta_i' = \Delta_i \qquad \text{for } i \neq -2, -1, 0, 1, \tag{31}$$

$$F\mathfrak{L}_i' = F\mathfrak{L}_i \qquad \text{for } i \neq 0, 1, \tag{32}$$

$$F\mathfrak{L}_{(i)}' = F\mathfrak{L}_{(i)} \qquad \text{for } i \neq 0, 1, 2. \tag{33}$$

We also have

$$\Delta_1' = 4\mathfrak{o}, \Delta_0' \supseteq 4\mathfrak{o}, \tag{34}$$

$$F\mathfrak{L}_{(2)}' \supseteq F\mathfrak{L}_{(2)}, \tag{35}$$

$$D_i' = \alpha_i^2 D_i, \text{ with } \alpha_i \neq 0 \text{ if } i \geqq 2. \tag{36}$$

We are now going to prove that l and L' satisfy the conditions of the

theorem. Our first task is to prove that l has a lower type than L'. Condition (7) must be verified for $i = 0, 1$; for $i = 0$, it is trivial since $\dim \mathfrak{I}_0 = 0$; if $\dim \mathfrak{L}_1 = \dim \mathfrak{I}_1 > 0$ we cannot have $d_1 D_1 \to 1$, hence $\dim \mathfrak{L}_1 > \dim \mathfrak{I}_1$, hence $\dim \mathfrak{I}_1 \leqq \dim \mathfrak{L}_1'$. Condition (8) is immediate when $i \geqq 2$; if $\dim \mathfrak{L}_1' = \dim \mathfrak{I}_1$, the 2-modular components of l and L' have the same dimension, hence $d_1 D_1 \to 1$. The first half of (9) follows from (31) and (32) when $i \geqq 2$; if $i = 1$, it follows from (34); if $\delta_0 = 2\mathfrak{o}$, we have $\delta_{-1} = 2\mathfrak{o}$ and so $\Delta_0' = \Delta_0 = 2\mathfrak{o}$ by II; if $i < 0$, $\delta_i \subseteq \Delta_i' + 2^{i+2}\mathfrak{o}$ since $\delta_i \subseteq 2\mathfrak{o}$. Similarly, the second part of (9) is immediate when $i \geqq 3$; consider $i = 2$ with $\dim \mathfrak{I}_2 = \dim \mathfrak{L}_2'$; then $\Delta_1' \subseteq \delta_1 + 8\mathfrak{o}$ if $\Delta_1' = \Delta_1$; so assume that $\Delta_1 \subseteq 8\mathfrak{o}$ and $\delta_1 \subseteq 8\mathfrak{o}$; this means that $\dim \mathfrak{L}_2 = \dim \mathfrak{I}_2$ with the 4-modular components of $\mathfrak{L}_2$ and $\mathfrak{I}_2$ either 0 or improper, and hence the dimensions of these components have the same parity; this means that the dimensions of the 2-modular components of $\mathfrak{L}_2$ and $\mathfrak{I}_2$ have opposite parity, and this denies (8); if $i = 1$ with $\Delta_0' = 2\mathfrak{o}$ and $\dim \mathfrak{I}_1 = \dim \mathfrak{L}_1'$, we see that $\dim \mathfrak{L}_1 - 1 = \dim \mathfrak{I}_1 > 0$ and $d_1 D_1 = d_1 D_1' \to 4$; hence $\Delta_0' = \Delta_0 \subseteq \delta_0 + 4\mathfrak{o}$ by (10); if $i = 0$ with $\dim \mathfrak{L}_0' = \dim \mathfrak{I}_0 = 0$, $\Delta_{-1}' \subseteq 2\mathfrak{o} \subseteq \delta_{-1} + 2\mathfrak{o}$; for $i < 0$, the condition is trivial. Hence (9) is completely verified. Condition (10) is immediate for $i \geqq 3$; consider $i = 2$ and suppose that $\dim \mathfrak{L}_2' - 1 = \dim \mathfrak{I}_2 > 0$ with $d_2 D_2' \to 2^3$; if $\Delta_1 = 4\mathfrak{o}$ or $\delta_1 = 4\mathfrak{o}$, we are through; so assume that $\Delta_1 \subseteq 8\mathfrak{o}$ and $\delta_1 \subseteq 8\mathfrak{o}$; the dimensions of the 4-modular components of $\mathfrak{L}_2$ and $\mathfrak{I}_2$ must therefore have the same parity, hence the same is true for the 2-modular components, hence $d_2 D_2 \to 1$, which contradicts $d_2 D_2' \to 2^3$; for $i = 1$, we have $\delta_0 = 2\mathfrak{o}$ if $\dim \mathfrak{I}_1 > 0$ since $\mathfrak{I}_{[0]} = 0$; for $i < 1$, the condition (10) is trivial. For $i \geqq 2$, (11) follows from (31), (32), (36); for $i = 1$, it follows from (34); for $i < 1$, it is vacuous since $\dim \mathfrak{I}_i$ is then 0. We have therefore proved that l has a lower type than L'.

We are now going to prove

$$\Delta_1' \to \mathfrak{L}_{(3)}/\mathfrak{I}_{[1]}, \tag{37}$$

$$\mathfrak{L}_{(3)}/\mathfrak{I}_{[1]} \cong FH(0) \text{ implies } \Delta_1'\delta_1 \subseteq \delta_1^2. \tag{38}$$

If $\Delta_1 = 4\mathfrak{o}$, the results follow from (33), (34), I and III; if $\delta_1 = 4\mathfrak{o}$, they follow from (33), (34), and II. We therefore assume that $\Delta_1 \subseteq 8\mathfrak{o}$ and $\delta_1 \subseteq 8\mathfrak{o}$; thus the dimensions of the 4-modular components of $\mathfrak{L}_2$ and $\mathfrak{I}_2$ have the same parity and $\mathfrak{I}_2 = \mathfrak{I}_{[1]}$. By (8) l, L', we have $\dim \mathfrak{I}_2 < \dim \mathfrak{L}_2' = \dim \mathfrak{L}_2$. First let $\Delta_1 = 8\mathfrak{o}$. We can then assume that $\dim \mathfrak{I}_{[1]} = \dim \mathfrak{I}_2 = \dim \mathfrak{L}_2' - 1 = \dim \mathfrak{L}_2 - 1$, and that the 8-modular component of L is 1-dimensional, in virtue of Proposition 16. Then the dimensions of the 2-modular components of l, L have the same parity so that

$\operatorname{ord}(d(\mathfrak{L}_{(3)}/\mathfrak{l}_{[1]}))$ is odd; here $F\mathfrak{l}_{[1]} \to F\mathfrak{L}_{(3)}$ by I; hence $\mathfrak{L}_{(3)}/\mathfrak{l}_{[1]}$ is of the form $F(\epsilon \oplus 2\delta)$, hence (37) and (38) follow. We now assume that $\Delta_1 = 0$; in this case, $\mathfrak{L}_{(3)} = \mathfrak{L}_2$. By Proposition 16, we need only verify (37) and (38) for $\dim \mathfrak{L}_2 - \dim \mathfrak{l}_2 = 1$ or 2. We still have $\mathfrak{l}_2 = \mathfrak{l}_{[1]}$. If $\dim \mathfrak{L}_2 - \dim \mathfrak{l}_2 = 1$, then (38) is vacuous; we prove (37); as before we see that the dimensions of the 2-modular components of $\mathfrak{L}_2$ and $\mathfrak{l}_2$ have the same parity, hence $\operatorname{ord}(d(\mathfrak{L}_{(3)}/\mathfrak{l}_{[1]}))$ is even; here $F\mathfrak{l}_{[1]} \to F\mathfrak{L}_{(3)}$ by I; hence $\Delta_1' \to \mathfrak{L}_{(3)}/\mathfrak{l}_{[1]}$. We now assume that $\dim \mathfrak{L}_2 = \dim \mathfrak{l}_2 + 2$. Using the above methods, we find that $\mathfrak{L}_{(3)}/\mathfrak{l}_{[1]}$ is of the form $F(\epsilon \oplus 2\delta)$ so that (37) and (38) follow.

Now we establish I-III for l, L'. For $i \geqq 2$, they are immediate. If $i = 1$, we get I from (37) and (33); if $i \leqq 0$, the proof of I is trivial since $\mathfrak{l}_{[i]}$ is then 0. If $i = 1$, II is trivial; if $i = 0$, it follows from (35); if $i = -1$, we can take $\delta_{-1} = 2\mathfrak{o}$, then $\delta_{-1} \to F\mathfrak{L}_{(1)}$ so that L has a 2-modular component, hence $\delta_{-1} \to F\mathfrak{L}_{(1)}'$; for $i \leqq -2$, II is trivial since δ_i is then 0. For $i = 1$, III follows from (38); if $i = 0$ and $\Delta_0' = 2\mathfrak{o}$, then $F\mathfrak{L}_{(2)}'$ is not an $FH(0)$; if $i \leqq -1$, the condition is trivial since $\Delta_i' \subseteq 2\mathfrak{o}$. This proves I-III. By (33), it suffices to prove IV for $i = -1, 0, 1$; for $i = -1, 0$ it is true since $\mathfrak{l}_i$ is then 0; for $i = 1$, it follows from (35). Similarly with V.

We have therefore proved that l, L' satisfy the conditions of the theorem. Now $\dim L' = \dim L$ and $\operatorname{len} L' < \operatorname{len} L$ since $\operatorname{len} L > 1$; hence by (16), we have $\Phi(L') < \Phi(L)$. Hence by the inductive assumption, we have $l \to L' \to L$.

When l_1 is unimodular. We shall divide the work into a number of cases as we did in the last section. All these cases have a certain computation in common which we now perform.

Remark 4. If $l_1 \to L_1$, then $l \to L$.

Proof. We write $L_1 = J \oplus L_1'$ with $J \cong l_1$ and $\dim L_1' \geqq 0$. If l_1 is proper and if L_1' is a $B(0)$, it is easily seen that we can find a new decomposition for L_1 of the above type and such that L_1 is an $H(0)$. So we can assume that

(39) $\qquad L_1'$ is not a $B(0)$ if $N(l_1) = \mathfrak{o}$.

If L_1' is a $B(\rho)$ with $N(l_1) = \mathfrak{o}$ and $N(L_2) = 4\mathfrak{o}$, referring to Remark 2 allows us to take a new decomposition for $L_1' \oplus L_2$ in which L_1' is a $B(0)$; utilizing (39), we can therefore assume that

(40) $\qquad L_1'$ is not a $B(\alpha)$ if $N(l_1) = \mathfrak{o}$, $N(L_2) = 4\mathfrak{o}$.

We define

$$L' = L_1' \oplus (\bigoplus_{\lambda \geqq 2} L_\lambda), \qquad l' = \bigoplus_{\lambda \geqq 2} l_\lambda.$$

It suffices to prove that $l' \to L'$.

First we establish that l' has a lower type than L'. It is evident that

$$\Delta_i' = \Delta_i, \qquad \delta_i' = \delta_i \qquad \text{for } i \geqq 0, \tag{41}$$

$$\dim \mathfrak{L}_i' - \dim \mathfrak{l}_i' = \dim \mathfrak{L}_i - \dim \mathfrak{l}_i \qquad \text{for all } i, \tag{42}$$

and that $d_i' = d_i$, $D_i' = D_i$ when $\mathfrak{l}_i' \neq 0$. Hence (7) and (8) hold for l', L'. The first half of (9) clearly holds for $i \geqq 0$; if $i \leqq -1$, it is true since δ_i' is then 0 or $2\mathfrak{o}$. The second half of (9) follows from (41), (42) when $i \geqq 1$; if $i = 0$, the condition on the dimensions implies that $\Delta_{-1}' \subseteq 2\mathfrak{o}$; if $i \leqq -1$, it is trivial. Both (10) and (11) are consequences of (41), (42) since the dimensional conditions imply that $i \geqq 1$. Hence l' has a lower type than L'.

Now we prove that l', L' satisfy I-V. It is easily seen that we have $F\mathfrak{l}_{[i]}' \to F\mathfrak{L}_{(i+2)}'$ and hence $\mathfrak{L}_{(i+2)}'/\mathfrak{l}_{[i]}' \cong \mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]}$ for any $i \geqq 0$. Using (41), we therefore see that I-III hold for $i \geqq 0$. Similarly, IV and V hold for $i \geqq 0$. For $i \leqq -1$, we have $\mathfrak{l}_{[i]}' = 0$ and I is then easily deduced. Since $\delta_i' = 0$ for $i \leqq -2$, it suffices to consider II for $i = -1$ with $\delta_{-1}' = 2\mathfrak{o}$; this means that l has a proper 2-modular component and so $\delta_0 = 2\mathfrak{o} = \delta_0'$; if $\dim l_1 = \dim L_1$, we have $\delta_0 \subseteq \Delta_0 + 4\mathfrak{o}$ by (9), hence L' has a proper 2-modular component, hence $\delta_{-1}' \to F\mathfrak{L}_{(1)}'$; if $\dim l_1 + 1 = \dim L_1$, the result follows in the same way using (11); let $\dim l_1 + 2 = \dim L_1$; then II is immediate if $\Delta_0 = 2\mathfrak{o}$; if $N(L_2) = 4\mathfrak{o}$ and $N(l_1) = \mathfrak{o}$, it follows from (40); if $N(L_2) = 4\mathfrak{o}$ with $N(l_1) = 2\mathfrak{o}$, then

$$\delta_{-1}' = \delta_{-1} \to \mathfrak{L}_{(1)}/\mathfrak{l}_{[-1]} = L_1/l_1 \cong FL_1' \cong \mathfrak{L}_{(1)}'/\mathfrak{l}_{[-1]}';$$

it is impossible to have $L_1' \cong B(\rho)$ when $N(L_2) \subseteq 8\mathfrak{o}$ and $\delta_{-1}' = 2\mathfrak{o}$, for we then have

$$2\mathfrak{o} = \delta_0 \to \mathfrak{L}_{(2)}/\mathfrak{l}_{[0]} \cong FL_1' \cong FB(\rho);$$

when $\dim L_1 - \dim l_1 \geqq 3$, the condition is immediate by Proposition 16. Hence we have proved II.

For $i \leqq -2$, the proof of III is trivial since we then have $\delta_i' = 0$, so we consider $i = -1$; we can then assume that $\Delta_{-1}' = \mathfrak{o}$, $\delta_{-1}' = 2\mathfrak{o}$; then L' cannot have a 2-modular proper component, else we would not have $F\mathfrak{L}_{(1)}' \cong \mathfrak{L}_{(1)}'/\mathfrak{l}_{[-1]}' \cong FH(0)$; hence $\mathfrak{L}_{(1)}' \cong L_1' \cong B(0)$; by (39) l_1 must be improper and so $\delta_{-1} = 2\mathfrak{o}$; but this implies that $\mathfrak{L}_{(1)}/\mathfrak{l}_{[-1]} \cong \mathfrak{L}_{(1)}'/\mathfrak{l}_{[-1]}' \cong FH(0)$

which contradicts the original III. The proof of IV is immediate for $i \leqq -1$ since I_i' is then 0. The same with V.

We have therefore proved that l' and L' satisfy the conditions of the theorem. Now $\dim L' < \dim L$ and $\operatorname{len} L' \leqq \operatorname{len} L$, so that $\Phi(L') < \Phi(L)$ by (15). By the induction, we then have $l' \to L'$, hence $l \to L$. q. e. d.

We now consider the various cases. By (7), we always have $\dim l_1 \leqq \dim L_1$. In the first three cases, we shall assume that l_1 is proper, so by (17) we have L_1 proper too.

Case 1. $N(L_2) = 2\mathfrak{o}$. Write $L_1 = J \oplus L_1'$ with $\dim J = \dim l_1$, $N(J) = N(l_1)$ and L_1' proper or 0; since L_1 has a canonical basis, we can do this. By Proposition 11, we have $J \cong l_1 \bmod 2$. In virtue of Proposition 14 and Remark 1, we are entitled to assume that $J \cong l_1 \bmod 4$. If $\dim L_1' > 0$, L_1' must represent a unit ϵ, hence 4ϵ, hence by Propositions 14 and 12, we have $J' \cong l_1$. Hence $l \to L$ by Remark 4. We now assume that $L_1' = 0$. If $N(L_3) = 4\mathfrak{o}$, using Propositions 14 and 12 with Remark 2 allows us to take $J \cong l_1$, and so $l \to L$ by Remark 4. We proceed in the same way if L_2 represents 4ϵ with ϵ a unit, only this time we use Remark 1 again.

To complete this case, we must consider $l_1 \cong L_1 \bmod 4$ with L_2 representing no number 4ϵ, and $N(L_3) \subseteq 8\mathfrak{o}$. By (4), this means that $\dim L_2 \leqq 2$. By I, we have $F(l_1 \oplus 2\epsilon) \to F(L_1 \oplus L_2)$; using a determinantal argument when $\dim L_2 = 2$, we see that $F(l_1 \oplus L_2') \cong F(L_1 \oplus L_2)$ for some proper 2-modular L_2' with $\dim L_2' \leqq 2$. Hence $l_1 \oplus L_2' \cong L_1 \oplus L_2$ by Theorem 9.1 of [8]. Using Remark 1, we can assume that $l_1 \cong L_1$. Hence $l \to L$ by Remark 4.

Case 2. $N(L_2) = 4\mathfrak{o}$. It follows from IV that

$$F((1 + 4\omega) \oplus l_1) \to F(1 \oplus L_1).$$

Now write $L_1 = L_1' \oplus \mathfrak{o}x_0$ with $x_0^2 = \epsilon$ a unit. Since L_2 represents 4δ with δ a unit, it is easily seen that $\mathfrak{o}x_0 \oplus L_2 = \mathfrak{o}\bar{x}_0 \oplus \bar{L}_2$ with $\bar{x}_0^2 = \epsilon(1 + 4\omega)$; put $\bar{L}_1 = L_1' \oplus \mathfrak{o}\bar{x}_0$; then the Jordan form $\bar{L}_1 \oplus \bar{L}_2 \oplus L_3 \oplus \cdots$ continues to satisfy the conditions of the theorem by Remark 2. But $1 \oplus L_1 \cong (1 + 4\omega) \oplus \bar{L}_1$ and so $Fl_1 \to F\bar{L}_1$, hence $l_1 \to \bar{L}_1$ by Theorem 2, hence $l \to L$ by Remark 4.

Case 3. $N(L_2) \subseteq 8\mathfrak{o}$. Using I, we see that $Fl_1 \to FL_1$, hence $l_1 \to L_1$ by Theorem 2, hence $l \to L$ by Remark 4.

We have covered all cases with l_1 proper unimodular. Now consider l_1 improper; then L_1 may or may not be improper; but it follows from (9) that L_1 is improper whenever $\dim l_1 = \dim L_1$.

Case 4. $N(L_2) = 2\mathfrak{o}$. Using the fact that $\dim l_1 < \dim L_1$ when $N(L_1) = \mathfrak{o}$, and referring to the decomposition of a unimodular lattice that is given in (12)-(14) of [9], it is easily seen that we can write $L_1 = J \oplus L_1'$ with $\dim J = \dim l_1$ and $N(J) = N(l_1)$; L_1' may be 0. Then $J \cong l_1 \bmod 2$ by Proposition 11. By Propositions 14 and 12, we can write $J \oplus L_2 = J' \oplus L_2'$ with $l_1 \cong J'$; by Remark 1, we can assume that actually $l_1 \cong J$. Hence $l \to L$ by Remark 4.

Case 5. $N(L_2) \subseteq 4\mathfrak{o}$. Since $\mathfrak{A}_{-1} \to \mathfrak{L}_{(1)}/\mathfrak{l}_{[-1]}$, we have $\mathfrak{A}_{-1} \to L_1/l_1$, hence $Fl_1 \to FL_1$. We know $N(l_1) \subseteq N(L_1)$. Apply Theorem 2 to l_1, L_1. Then

$$\mathfrak{A}_{-1}(L_1) = \mathfrak{A}_{-1} \text{ or } 0, \qquad \mathfrak{d}_{-1}(l_1) = 0;$$

hence the first condition of Theorem 2 is satisfied; the second one is trivial. Hence $l_1 \to L_1$. Hence $l \to L$ by Remark 4.

When l_1 is 2-modular improper. We continue on the same lines as in the last two sections. First we perform some computations, then we apply them to the different cases that follow.

Remark 5. Let m be a binary component of l_1 and let M be a binary component of L_1 or L_2. If $m \to M$, then $l \to L$.

Proof. Clearly m is either a $2 \circ H(0)$ or a $2 \circ H(\rho)$. If M is 2-modular, then $m \cong M$. If M is unimodular, then M is a $B(0)$, $B(\rho)$ or $H(0)$. Let $L_a = M \oplus L_a'$; a is then 1 or 2. Put $L_\lambda' = L_\lambda$ for $\lambda \neq a$ and define L' by the Jordan form $L' = \oplus L_\lambda'$. Define l' in the same way by putting $l_1 = m \oplus l_1'$. We see that

$$\mathfrak{d}_i' = \mathfrak{d}_i \text{ for all } i, \qquad \mathfrak{A}_i' = \mathfrak{A}_i \text{ for } i \geqq 0, \tag{43}$$

$$\mathfrak{d}_0 = 0 \text{ or } 4\mathfrak{o}, \qquad \mathfrak{d}_i = 0 \text{ for } i \leqq -1. \tag{44}$$

We also have $\dim \mathfrak{L}_i' - \dim \mathfrak{l}_i' = \dim \mathfrak{L}_i - \dim \mathfrak{l}_i$ when $i \geqq 1$. Also $d_i = 4d_i'$, $D_i = \alpha D_i'$, with $\alpha = 1$ or 4, when $\mathfrak{l}_i' \neq 0$. Using this information, it is clear that l' has a lower type than L'. Now,

$$\mathfrak{L}_{(i+2)}'/\mathfrak{l}_{[i]}' \cong \mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]} \qquad \text{if } i \geqq 0. \tag{45}$$

Using (43), (44), (45), it is evident that I-III hold for l', L'. Similarly, IV and V. We omit the details as quite straight-forward; there are none of the technical difficulties that we encountered with this kind of proof earlier on. Hence l', L' satisfy the conditions of the theorem. But $\dim L' < \dim L$,

$\operatorname{len} L' \leqq \operatorname{len} L$; hence $\Phi(L') < \Phi(L)$ by (15), hence $l' \to L'$ by the induction, hence $l \to L$. q. e. d.

Remark 6. *If* $FB(\alpha) \to FL_1$, then $l \to L$.

Proof. If L_1 is proper, using Theorem 2, we see that $B(\alpha) \to L_1$ so that $J \cong B(\alpha)$ is a component of L_1. If L_1 is improper, α must be 0 when $\dim L_1 = 2$ so that $J \cong H(0)$ is a component of L_1; for higher dimensions, the last remark is always true. In both cases, we can therefore write $L_1 = J \oplus L_1'$ with $J \cong B(\alpha)$ or $H(0)$. Define $K \subset J$ such that $K \cong 2 \circ H(\beta)$. Apply Remark 3. Then (21)-(24) are easily verified, hence $l \to L$. q. e. d.

Case 1. $N(L_2) = 2\mathfrak{o}$. First let $\dim L_1 \geqq 2$. Using Table I, equation (4), and Propositions 11 and 14, we see that $L_1 \oplus L_2$ has another Jordan form $L_1' \oplus L_2'$ with L_1' representing one of $H(0)$, $B(0)$, $B(\rho)$. By Remark 1, we know that $\oplus L_\lambda'$, $\oplus l_\lambda$ satisfy the conditions of the theorem. Hence $l \to L$ by Remark 6. Now let $\dim L_1 = 1$ so that L_1 is proper. If $\dim L_2 \geqq 3$, it is easily seen that we can write $L_1 \oplus L_2$ in the Jordan form $L_1' \oplus L_2'$ with L_2' representing either $2 \circ H(0)$ or, if we wish, $2 \circ H(\rho)$. By Remark 1, $\oplus L_\lambda'$ and $\oplus l_\lambda$ satisfy the conditions of the theorem. By Remark 5, we have $l \to L$. We cannot have $\dim L_1 = 1$ with $\dim L_2 \leqq 2$ in virtue of (8) and (10).

Case 2. L_2 improper 2-modular. First let L_1 be proper. It is easily seen that we can write $L_1 \oplus L_2 = L_1' \oplus L_2'$ with $2 \circ H(0) \to L_2'$ or, if we wish, $2 \circ H(\rho) \to L_2'$. By Remark 2, $\oplus L_\lambda'$ and $\oplus l_\lambda$ continue to satisfy the conditions of the theorem. By Remark 5, we therefore have $l \to L$. We now take L_1 improper. By Remark 6, we can take $L_1 \cong H(\rho)$; by Remark 5, we can take $L_2 \cong 2 \circ H(\alpha)$, $l_1 \cong 2 \circ H(\beta)$ with $\alpha = 0$ or ρ, $\beta = 0$ or ρ, and $\alpha \neq \beta$. If $N(L_3) = 4\mathfrak{o}$, it follows from Propositions 11, 12 and 14 that $L_2 \oplus L_3 = L_2' \oplus L_3'$ with $L_2' \cong 2 \circ H(\beta)$; it is easily verified that the new Jordan forms satisfy the conditions of the theorem. (None of the earlier technical difficulties arise in dong this.) Hence $l \to L$ by Remark 5. We therefore assume that $N(L_3) \subseteq 8\mathfrak{o}$. By I, we have $Fl_{[0]} \to F\mathfrak{L}_{(2)}$, hence $F(2 \circ H(\beta)) \to FH(\rho) \oplus F(2 \circ H(\alpha))$, hence $FB(\beta) \to FH(\rho) \oplus FB(\alpha)$. If $\alpha = \rho$, we have $\beta = 0$, hence $FB(0)$ is isotropic but $FH(\rho) \oplus FB(\rho)$ is not; so this is impossible. If $\alpha = 0$, we have $\beta = \rho$ and $FB(\rho) \oplus FH(0) \cong FH(\rho) \oplus FB(0)$, and this is impossible by Table II and Proposition 10.

Case 3. $L_2 \cdot L_2 \subseteq 4\mathfrak{o}$. First let L_1 be proper. If $N(L_2) \subseteq 8\mathfrak{o}$, then $Fl_1 \to FL_1$ by I. If $N(L_2) = 4\mathfrak{o}$, we have $F((1 + 4\omega) \oplus l_1) \to F(1 \oplus L_1)$

by V. Put $L_1 = L_1' \oplus \mathfrak{o}x_0$ with $x_0^2 = \epsilon$ where ϵ is a unit. Then $\mathfrak{o}x_0 \oplus L_2 = \mathfrak{o}\bar{x}_0 \oplus \bar{L}_2$ with $\bar{x}_0^2 = \epsilon(1+4\omega)$. Put $\bar{L}_1 = L_1' \oplus \mathfrak{o}\bar{x}_0$. Then $\bar{L}_1 \oplus \bar{L}_2 \oplus L_3 \oplus \cdots$ and $\oplus l_\lambda$ still satisfy the conditions of the theorem by Remark 2. Since $(1) \oplus L_1 \cong (1+4\omega) \oplus \bar{L}_1$, this implies that $Fl_1 \to F\bar{L}_1$. So we take it that $Fl_1 \to FL_1$ when L_1 is proper. Hence $FB(\alpha) \cong F(2 \circ H(\alpha)) \to FL_1$, hence $l \to L$ by Remark 6. Now let L_1 be improper. By Remark 6, we can assume that $L_1 \cong H(\rho)$. So $l_1 \cong 2 \circ H(\alpha)$ with $\alpha = 0$ or ρ. By V, we then have $F((1+4\omega) \oplus B(\alpha)) \to F(1 \oplus H(\rho))$; this is impossible since the first of these spaces is isotropic while the second is not.

We have now completed the proof of Theorem 3.

IV. Practical Aspects.

The central feature in all our tests (Theorems 1-3) is the notion of space representation. It is therefore important to be able to determine when $U \to V$. Solutions are available in the literature ([6], [12]), so we merely give a summary here. Consider the quadratic spaces U, V over the local field F. Put $\mathrm{def} = \dim V - \dim U$ and assume that $\mathrm{def} \geqq 0$. Let SU, SV denote Hasse symbols of U, V. Then for spaces of the same dimension,

$$U \cong V \Leftrightarrow dU = dV, SU = SV.$$

And $U \to V$ if and only if

$\mathrm{def} = 0$: $\quad U \cong V,$

$\mathrm{def} = 1$: $\quad U \oplus (dU \cdot dV) \cong V,$

$\mathrm{def} = 2$: $\quad dU \cdot dV = -1$ implies $U \oplus FH(0) \cong V,$

$\mathrm{def} \geqq 3$: $\quad$ none.

This gives us all we need to carry out the test in Theorem 1. We therefore assume that $2 \in F$ is a prime. We have to answer the following questions: when is $V/U \cong W$? When is $\alpha \to V/U$? When is $\mathfrak{p} \to V/U$? The answers to the first two questions are $U \oplus W \cong V$, and $(\alpha) \oplus U \to V$, respectively. We consider when $\mathfrak{p} \to V/U$. We can assume that $\mathfrak{p} \neq 0$, $\mathrm{def} > 0$, and $U \to V$. If $\dim V/U = 1$, we have $\mathfrak{p} \to V/U$ if and only if $\mathrm{ord}\,\mathfrak{p}$ and $\mathrm{ord}(dV \cdot dU)$ have the same parity. If $\dim V/U \geqq 2$, we have $\mathfrak{p} \to V/U$ if and only if

$$U \oplus FB(\rho) \cong V \text{ implies } \mathrm{ord}\,\mathfrak{p} \text{ even},$$

$$U \oplus FH(\rho) \cong V \text{ implies } \mathrm{ord}\,\mathfrak{p} \text{ odd}.$$

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ON POWER MULTIPLICATIVE NORMS.*

By SILVIO AURORA.

1. Introduction. In [4], A. Ostrowski classified the absolute values (for definitions, see section 2) of fields as either archimedean or non-archimedean, and showed that a field with an archimedean absolute value may be embedded in the complex field.

The key result of this paper occurs in seceion 4 and describes the structure of norms[1] (of rings) having the properties that $\| x^2 \| = \| x \|^2$ for all x, and $\| \cdots xy \cdots \| = \| \cdots yx \cdots \|$ for all x and y. Such norms are classified as archimedean or non-archimedean in section 5, and the archimedean case is further broken down into two cases: the strictly archimedean, and the partially archimedean. In section 6, Ostrowski's embedding theorem is generalized to show that division rings having an archimedean norm of this type are algebraically embeddable in the ring of quaternions. Also, a division ring with a non-discrete, strictly archimedean norm of this type is shown to be algebraically and topologically isomorphic to a subring of the ring of quaternions. Finally, section 7 outlines a description of all norms of the ring of integers or the field of rationals such that $\| x^2 \| = \| x \|^2$ for all x.

2. Pseudonorms and pseudometric rings. We shall consider in this paper associative rings which do not necessarily possess a unit element.

By a *pseudonorm* of a ring R is meant a real-valued function N defined on R and such that:

(i) $N(0) = 0$.

(ii) $N(x+y) \leqq N(x) + N(y)$ for all x and y in R.

(iii) $N(-x) = N(x)$ for all x in R.

(iv) $N(xy) \leqq N(x)N(y)$ for all x and y in R.

(We find it more convenient to use standard function notation, such as $N(x)$, rather than the usual notation for norms, such as $\| x \|$.) A pseudonorm N

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[1] Many of the results are actually stated for the more general case of *pseudonorms*.

such that $N(x) = 0$ only if $x = 0$ is called a *norm*. If a pseudonorm (norm) N satisfies the condition: (iv') $N(xy) = N(x)N(y)$ for all x and y in R, which is a stronger requirement than (iv), then N is called a *pseudo absolute value* (*absolute value*).

If N is a pseudonorm for a ring R, an element x of R is called N-*null* provided that $N(x) = 0$; the set of N-null elements of R is a two-sided ideal $I(N)$, which we shall call the *null ideal* of N. It is clear that the function N is constant on each residue class modulo $I(N)$, so that the function $\bar{N}$ such that $\bar{N}(A)$ is the common value assumed by N on the elements of the residue class A is a natural norm for the ring $\bar{R} = R/I(N)$ of residue classes modulo $I(N)$. (Compare Bourbaki in Exercise 13 at the end of Chapter IX, § 3 of [3], where $\bar{N}$ is said to be "associated" with N.)

We now exhibit some examples of pseudonorms:

(1) The *zero pseudonorm*, Z, of any ring is defined to be identically zero on the ring. This is always a pseudo absolute value, and it is clear that a pseudonorm N coincides with Z if and only if $I(N)$ is the entire ring.

(2) The *trivial norm*, T, of any ring is defined to be identically 1 in value on the non-zero elements of that ring. This norm is an absolute value if and only if the ring has no proper zero-divisors.

(3) The *ordinary* absolute value, A, is defined by $A(x) = |x|$ for all x, on any subring of the ring of real quaternions.

(4) The *p-adic* absolute value, B_p, where p is a rational prime, is defined on any subring of the field of rational numbers by the condition $B_p(x) = p^{-r}$ if x is a non-zero rational number having a representation $x = p^r \cdot (m/n)$, with m and n non-zero integers relatively prime to p, and with r an integer.

(5) If Q is a compact Hausdorff space and $C(Q)$ is the ring of all continuous complex-valued functions on Q, then the *supremum norm* S, such that $S(x) = \sup\{|x(q)| \mid q \text{ in } Q\}$, is of course a norm for $C(Q)$, but is an absolute value only when Q reduces to a single point.

(6) If I is a two-sided ideal of the ring R, the function $T^{(I)}$ which is the characteristic function of the complement of I is clearly a pseudonorm. Also, $T^{(I)}$ is a norm if and only if I is the zero ideal, and $T^{(I)}$ is a pseudo absolute value if and only if I is such that xy in I implies x in I or y in I.

The function $d(x, y) = N(x - y)$ is a pseudometric (metric) for R if

N is a pseudonorm (norm) for R. Also, the pseudometric $d(x,y)$ induces in the usual way a topology in R relative to which the ring operations are continuous. A ring R, together with a pseudonorm (norm, absolute value) for R, is called a *pseudometric ring* (*metric ring, ring with absolute value*). Thus, a pseudometric ring is a topological ring in the weak sense of the term; that is, the topology is not necessarily Hausdorff. In particular, a metric ring is a topological ring in the usual sense, and any pseudometric ring may be "coagulated" into a metric ring by passing to the ring of residue classes modulo the null ideal, with the "associated" norm. Such concepts as fundamental sequences, completeness, etc., are introduced in the obvious way for metric rings, and the usual results about these concepts are easily established.

A topological ring R such that $R^2 \neq \{0\}$ will be called *simple* if the only closed two-sided ideals are $\{0\}$ and R. In particular, for a discrete ring, simplicity coincides with the usual, purely algebraic concept as defined in [1]. For a pseudometric ring, the null ideal is obviously a closed two-sided ideal; thus, the null ideal of a simple pseudometric ring is either $\{0\}$, in which case the pseudonorm is a norm, or is the whole ring, in which case the pseudonorm is Z. That is, a simple pseudometric ring is either a simple metric ring or has the zero pseudonorm. It is clear that a metric ring which is simple in the purely algebraic sense is a simple metric ring. However, the converse is not true, for the ring generated by π and the rational numbers, and having A as its norm, is a simple metric ring but is obviously not simple in the purely algebraic sense.

3. The semi-lattice of pseudonorms. In this section, we shall consider an ordering of the set $\mathfrak{N}_R$ of all pseudonorms of a ring R. If N and N' are pseudonorms for R, we say that N' is *subordinate* to N, and write $N' \leqq N$ (also $N \geqq N'$), provided that $N'(x) \leqq N(x)$ for all x in R. This relation is clearly a partial ordering of $\mathfrak{N}_R$. It may also be noted that if $N' \leqq N$, then $I(N) \subset I(N')$. However, the correspondence which assigns $I(N)$ to N is not one-to-one. For example, T is subordinate to A, but distinct from A, on the ring of integers, while $I(A) = I(T)$ since both ideals contain only 0.

A non-empty family $\{N_\lambda \mid \lambda \in \Lambda\}$ of pseudonorms of a ring R is said to be *bounded* if, for each x in R, the set $\{N_\lambda(x) \mid \lambda \in \Lambda\}$ is bounded. If $\{N_\lambda \mid \lambda \in \Lambda\}$ is a bounded family of pseudonorms of R, then the function N such that, for each x in R, $N(x) = \sup\{N_\lambda(x) \mid \lambda \in \Lambda\}$ is easily seen to be a pseudonorm on R; we shall call N the *join* of the family $\{N_\lambda \mid \lambda \in \Lambda\}$ and denote it by the symbol $\bigvee\{N_\lambda \mid \lambda \in \Lambda\}$. (For a finite family $\{N_1, \cdots, N_k\}$, we shall also denote the join by $N_1 \vee \cdots \vee N_k$.) In terms of the cus-

tomary terminology for partially ordered sets, the bounded families of pseudonorms of R are precisely the subsets of $\mathfrak{N}_R$ which have upper bounds in $\mathfrak{N}_R$, and the join of a bounded family is its least upper bound in $\mathfrak{N}_R$. Since a finite family of pseudonorms of R is always bounded and consequently has a join in $\mathfrak{N}_R$, the system $\mathfrak{N}_R$ is an upper semi-lattice.

A pseudonorm N of a ring R will be called *stable* if $N(\cdots xy\cdots) = N(\cdots yx\cdots)$ for all x and y in R. Any pseudo absolute value is clearly stable, and all pseudonorms of a commutative ring are stable. However, if R is any ring having elements x and y such that $xy = 0$ but $yx \neq 0$, then $T(xy) = 0$ and $T(yx) = 1$, so T is not stable on such a ring; thus, there even exist norms which are not stable.

If a pseudonorm N of a ring R has the property that $N(x^2) = [N(x)]^2$ for all x in R, then N will be called a *power multiplicative* pseudonorm (or *PM* pseudonorm for brevity). It is easily seen that if N is a *PM* pseudonorm then $N(x^r) = [N(x)]^r$ for every x in R and every natural number r; it follows easily that $N(x^r \cdot x^s) = N(x^r)N(x^s)$ for any x in R when r and s are natural numbers. The latter property explains the choice of the term "power multiplicative" to describe such a pseudonorm. Any pseudo absolute value is a *PM* pseudonorm, and the supremum norm S (of example (5) above) is also clearly power multiplicative. Not all pseudonorms are power multiplicative, however, even if they are norms; for example, the functions $2A$ and $A + T$ are norms of the real field but are not power multiplicative.

It is easily shown that the join of stable pseudonorms is stable, and the join of *PM* pseudonorms is a *PM* pseudonorm. Since a pseudo absolute value is always a stable *PM* pseudonorm, it follows that the join of a bounded family of pseudo absolute values is a stable *PM* pseudonorm. The converse, that every stable *PM* pseudonorm is the join of a bounded family of pseudo absolute values, is also true and constitutes the fundamental theorem on stable *PM* pseudonorms. This theorem will be established in the following section.

4. The structure of stable *PM* pseudonorms. We shall first introduce a method for constructing from a given pseudonorm various other pseudonorms subordinate to it. Let N be a pseudonorm of a ring R, and let c be an element of R which is not N-null. For any x in R, we have the inequalities:

$$N(x) \geqq N(xc)/N(c) \geqq N(xc^2)/[N(c)]^2 \geqq \cdots \geqq 0.$$

Then for any x in R, there is defined a non-negative number $N_c(x) = \lim_{r\to\infty} N(xc^r)/[N(c)]^r$. If N is a stable pseudonorm of R, it is easily

proved that N_c is a stable pseudonorm subordinate to N. We shall refer to N_c as *the subordinate pseudonorm of N induced by c*, or simply as an *induced pseudonorm* of N. In the present discussion, we are concerned mainly with power multiplicative pseudonorms; the lemma which follows gives some properties of the induced pseudonorms for this special case.

LEMMA 1. *Let N be a non-zero stable PM pseudonorm of a ring R, and let c be an element of R which is not N-null. Then N_c is a non-zero stable PM pseudonorm of R such that:* (i) *N_c is subordinate to N,* (ii) $N_c(c) = N(c)$, (iii) $N_c(cx) = N_c(c)N_c(x)$ *for all x in R.*

Proof. That N_c is a stable pseudonorm subordinate to N has already been noted. We have $N_c(c) = \lim_{r\to\infty} N(c^{r+1})/[N(c)]^r = \lim_{r\to\infty} [N(c)]^{r+1}/[N(c)]^r = N(c)$ since N is power multiplicative. This proves (ii) and also shows that N_c is non-zero, since $N_c(c) = N(c) \neq 0$. To prove (iii), we note that $N_c(cx) = \lim_{r\to\infty} N(c^{r+1}x)/[N(c)]^r = N(c) \lim_{r\to\infty} N(c^{r+1}x)/[N(c)]^{r+1} = N(c)N_c(x) = N_c(c)N_c(x)$ for any x. Finally, to show that N_c is power multiplicative, we note that $\{N_c(x)\}^2 = \{\lim_{r\to\infty} N(xc^r)/[N(c)]^r\}^2 = \lim_{r\to\infty} \{N(xc^r)/[N(c)]^r\}^2 = \lim_{r\to\infty} N(x^2c^{2r})/[N(c)]^{2r} = N_c(x^2)$ for any x.

A non-empty set $\mathfrak{N}$ of non-zero stable PM pseudonorms on a ring R will be called a *hereditary system* if $\mathfrak{N}$ contains all the induced pseduonorms of its members. The preceding lemma tells us that the set of all non-zero stable PM pseudonorms on a ring is hereditary if it is non-empty, and the set of all non-zero stable PM pseudonorms subordinate to a given non-zero stable PM pseudonorm is always hereditary.

LEMMA 2. *A minimal element of a hereditary system $\mathfrak{N}$ must be a non-zero pseudo absolute value.*

Proof. If N is a minimal element of $\mathfrak{N}$, then N is non-zero since all elements of $\mathfrak{N}$ are non-zero. If c and d are arbitrary elements of the ring involved, we have $N(cd) = N(c)N(d)$ if c is N-null, for both sides of the equation are then zero. If c is not N-null, then N_c is defined and belongs to $\mathfrak{N}$, with $N_c(cd) = N_c(c)N_c(d)$ by Lemma 1(iii). But $N_c \leqq N$, so $N = N_c$ since N is a minimal element of $\mathfrak{N}$; thus, $N(cd) = N(c)N(d)$ in this case also. This shows that N is a pseudo absolute value.

It should be noted that Lemma 2 describes the minimal elements of a hereditary system *if there are minimal elements in it.* The example which follows shows that a hereditary system need not have minimal elements. In

the ring R of even integers, let $N^n(x) = [B_2(x)]^n$ for each x in R. Then $\mathfrak{N} = \{N^1, N^2, N^3, \cdots\}$ is a hereditary system and is simply ordered, with $N^1 \geqq N^2 \geqq N^3 \geqq \cdots$; also, $N^n(2) = 2^{-n}$ and $N^{n+1}(2) = 2^{-n-1}$, so $N^n \neq N^{n+1}$ for every natural number n, and it follows that $\mathfrak{N}$ has no minimal element. The same example shows that a pseudo absolute value in a hereditary system is not necessarily a minimal element of that system, for in this example, it is easily seen that every element of $\mathfrak{N}$ is actually an absolute value.

We shall now consider certain collections of pseudonorms on a ring R that form hereditary systems which always have minimal elements. If N is a non-zero stable PM pseudonorm on R, and if c is an element of R which is not N-null, let $[R;N;c]$ be the set of all stable PM pseudonorms N' on R such that: (i) N' is subordinate to N, (ii) $N'(c) = N(c)$, and (iii) $N'(cx) = N'(c)N'(x)$ for all x in R. Note that (ii) insures that the elements of $[R;N;c]$ are non-zero.

LEMMA 3. *$[R;N;c]$ is a hereditary system.*

Proof. $[R;N;c]$ is not empty since Lemma 1 implies that it contains N_c. If N' is any member of $[R;N;c]$ and d is not N'-null, then N'_d is a non-zero stable PM pseudonorm by Lemma 1. Also, N'_d is subordinate to N' and hence to N. Next,

$$N'_d(c) = \lim_{r\to\infty} N'(cd^r)/[N'(d)]^r = \lim_{r\to\infty} N'(c)N'(d^r)/N'(d^r) = N'(c) = N(c)$$

since N' is power multiplicative and has properties (iii) and (ii) of a system $[R;N;c]$. Finally,

$$\begin{aligned} N'_d(cx) &= \lim_{r\to\infty} \{N'(cxd^r)/[N'(d)]^r\} \\ &= \lim_{r\to\infty} \{N'(c)N'(xd^r)/[N'(d)]^r\} = N'(c)N'_d(x) = N'_d(c)N'_d(x) \end{aligned}$$

for any x. Thus, N'_d is also in $[R;N;c]$ when N' is in $[R;N;c]$, and the set $[R;N;c]$ is therefore a hereditary system.

LEMMA 4. *$[R;N;c]$ contains minimal elements.*

Proof. Let $\mathfrak{N}$ be any non-empty simply ordered subset of $[R;N;c]$. Let $N'(x) = \inf\{N_\alpha(x) \mid N_\alpha \in \mathfrak{N}\}$ for each x in R. It is easily verified that N' is a stable PM pseudonorm which satisfies (i), (ii) and (iii); thus, N' belongs to $[R;N;c]$. Clearly, N' is subordinate to every element of $\mathfrak{N}$. Thus, every non-empty simply ordered subset of $[R;N;c]$ has a lower bound in $[R;N;c]$. An application of Zorn's Lemma shows that $[R;N;c]$ contains minimal elements.

The preceding lemma shows that any system $[R;N;c]$ contains minimal elements, which are necessarily non-zero pseudo absolute values by Lemmas 2 and 3, and which are subordinate to N by the definition of $[R;N;c]$. If N is a pseudonorm of a ring R, and if c is an element of R which is not N-null, by a *component of N at c* we shall mean any pseudo absolute value N' subordinate to N and such that $N'(c)=N(c)$. (Thus, a component is always non-zero.) If N is a non-zero stable PM pseduonorm of a ring R and c is an element of R which is not N-null, then any component of N at c must obviously belong to $[R;N;c]$; on the other hand, the minimal elements of $[R;N;c]$ are clearly components of N at c, so there are always components of N at c when c is not N-null.

THEOREM 1. (Fundamental theorem) *If N is a non-zero stable PM pseudonorm on a ring R, then N is the join of a bounded family of pseudo absolute values on R.*

Proof. Let $\mathfrak{N}$ be the set of all minimal elements of the sets $[R;N;c]$ as c ranges over the elements of R which are not N-null. Then $\mathfrak{N}$ is non-empty by Lemma 4, and consists of components of N according to the discussion preceding this theorem. The elements of $\mathfrak{N}$ are all subordinate to N, so their join, N', is also subordinate to N.

If c is an N-null element of R, we have $N(c)=N'(c)=0$. If c is not N-null, then $[R;N;c]$ contains a minimal element N'' by Lemma 4, and $N''(c)=N(c)$ by definition of $[R;N;c]$; but since $N\geqq N'\geqq N''$, we must have $N(c)\geqq N'(c)\geqq N''(c)$, so $N(c)=N'(c)$. Thus, for any c in R, $N(c)=N'(c)$, whence $N=N'$. This proves that N is the join of the elements of $\mathfrak{N}$, which are components and, in particular, pseudo absolute values.

If $\mathfrak{N}'$ is any non-empty set of pseudo absolute values subordinate to N, then the join of $\mathfrak{N}'$ is clearly subordinate to N. On the other hand, if $\mathfrak{N}'$ is a bounded family of pseudo absolute values on R such that $\mathfrak{N}'\supset\mathfrak{N}$, then N, which is the join of $\mathfrak{N}$, is obviously subordinate to the join of $\mathfrak{N}'$. Thus, if $\mathfrak{N}'$ is a set of pseudo absolute values subordinate to N such that $\mathfrak{N}'\supset\mathfrak{N}$, then the join of $\mathfrak{N}'$ is precisely N. This hypothesis is clearly fulfilled if we take as $\mathfrak{N}'$ either the set of all components of N, or the set of all pseudo absolute values subordinate to N. We thus obtain the following corollaries.

COROLLARY 1. *A non-zero stable PM pseudonorm is the join of its set of components.*

COROLLARY 2. *A non-zero stable PM pseudonorm is the join of the set of all pseudo absolute values subordinate to it.*

It is also possible to describe the rings which admit non-zero stable *PM* pseudonorms.

COROLLARY 3. *In order that a ring R admit a non-zero stable PM pseudonorm, it is necessary and sufficient that there exist a two-sided ideal I contained properly in R and such that xy in I implies x in I or y in I.*

Proof. The sufficiency is seen by taking the pseudonorm $T^{(I)}$ of example (6) in section 2, for this pseudonorm is then a non-zero pseudo absolute value and therefore certainly a stable *PM* pseudonorm. To see the necessity, a non-zero stable *PM* pseudonorm has a non-zero component because of Corollary 1. and the null ideal of such a component may be taken as *I*.

An example of the representation of a stable *PM* pseudonorm in terms of pseudo absolute values is given by example (5) of section 2. For a fixed q in Q, let $S_q(x) = |x(q)|$ for all x in $C(Q)$. It is clear that S_q is a pseudo absolute value for each q, and that S_q is in fact a component of S. Clearly, $S = \vee\{S_q \mid q \in Q\}$, but the representation of S as the join of pseudo absolute values is not unique since we also have $S = \vee\{S_q \mid q \in E\}$ for any set E dense in Q.

In this example, it is also seen that even though the pseudonorm S is actually a norm, the representation of S is in terms of pseudo absolute values, but not absolute values. Indeed, $C(Q)$ admits no absolute values if Q has more than one point, for in this case $C(Q)$ clearly has proper zero-divisors, whereas a ring with absolute value obviously has no proper zero-divisors. However, for simple metric rings the representation of a stable *PM* norm can be shown to be in terms of absolute values.

For convenience, by a *PMN ring* we shall mean a metric ring having a stable *PM* norm. Also, terminology relating to the pseudonorm of a pseudometric ring will be applied to the ring itself. For example, by a "subordinate pseudonorm" of a pseudometric ring will be meant one subordinate to the pseudonorm of that ring.

LEMMA 5. *A subordinate pseudonorm of a pseudometric ring R is continuous on R.*

Proof. Let N be the pseudonorm of R, and let N' be a subordinate pseudonorm. The inequality $|N'(x) - N'(y)| \leqq N'(x-y) \leqq N(x-y)$ for all x and y in R implies that N' is continuous on R.

THEOREM 2. *Every non-zero subordinate pseudonorm (pseudo absolute value) of a simple metric ring is a norm (absolute value).*

Proof. By Lemma 5, a subordinate pseudonorm is continuous, so its null ideal is a closed two-sided ideal (different from the whole ring when the pseudonorm is non-zero). For a simple metric ring, it follows that the null-ideal of a non-zero subordinate pseudonorm N' is the zero ideal, whence N' is a norm. If N' was also a pseudo absolute value, then N' must be an absolute value.

If we apply this result to Corollaries 1 and 2 of Theorem 1, we may conclude that, for a simple metric ring having a stable PM norm, we have a representation of the norm in terms of absolute values.

COROLLARY 1. *The norm of a simple PMN ring is the join of its components, which are all absolute values.*

COROLLARY 2. *The norm of a simple PMN ring is the join of all the subordinate absolute values.*

5. The classification of stable *PM* pseudonorms. In the sequel, all rings are assumed to possess a unit element, e. A pseudonorm N on a ring R will be called *non-archimedean* if $N(x+y) \leqq \max(N(x), N(y))$ for all x and y in R, and N will be called *archimedean* if it is not non-archimedean.

If N is a pseudonorm of a ring R, and if γ is a positive number, let $N^\gamma(x) = [N(x)]^\gamma$ for all x in R. It is clear that N^γ satisfies all the defining conditions of a pseudonorm, except possibly (ii). Also, if N^γ is a pseudonorm, then it is easily proved that N^δ is a pseudonorm whenever $0 < \delta < \gamma$. If we define N^0 by the conditions $N^0(x) = 0$ when $N(x) = 0$ and $N^0(x) = 1$ when $N(x) \neq 0$, then N^0 is always a pseudonorm. Thus, the set of non-negative numbers γ such that N^γ is a pseudonorm on R must be a finite closed interval $[0, \mu]$, with $\mu \geqq 1$, or the infinite interval $[0, +\infty[$. It is easily proved that the infinite interval occurs if and only if N is non-archimedean. (Compare the corresponding statement in Chap. IX, § 3, no. 2 of [3] for the case of absolute values of division rings.) However, this characterization of the non-archimedean pseudonorms does not seem to be useful way of determining whether a given pseudonorm is non-archimedean.

We shall give below some criteria for classifying stable PM pseudonorms as archimedean or non-archimedean. First, we note that if N is a non-zero PM pseudonorm, then e is not in $I(N)$, so $N(e) = 1$ since $N(e) = N(e^2) = [N(e)]^2$. For a non-archimedean PM pseudonorm N, it follows that

$N(ne) \leqq 1$ for every natural number n. Thus, a *PM* pseudonorm N must be archimedean if there exists a natural number n for which $N(ne) > 1$.

LEMMA 6. *Let N be a PM pseudonorm, and let m be an integer with $m > 1$. Then $N(ne) \leqq 1$ for every natural number n if and only if $N(me) \leqq 1$.*

Proof. The result is easily proved along standard lines, by writing n^r in terms of the base m, etc.

LEMMA 7. *Let N be a PM pseudonorm of a ring R, and let m be an integer with $m > 1$. Then $N(me) \leqq 1$ if and only if*

$$N(x+y) \leqq \max(N(x), N(y))$$

whenever x and y are elements of R which commute.

Proof. First, suppose $N(x+y) \leqq \max(N(x), N(y))$ whenever x and y commute. Since N is a *PM* pseudonorm, we have $N(e) \leqq 1$. Let k be the smallest natural number for which $N((k+1)e) > 1$. But ke and e commute whenver k is a natural number, so

$$N((k+1)e) = N(ke+e) \leqq \max(N(ke), N(e)) \leqq 1,$$

and this is a contradiction. It follows that we have $N(ne) \leqq 1$ for every natural number n, so we have $N(me) \leqq 1$.

Conversely, if $N(me) \leqq 1$, then Lemma 6 implies that $N(ne) \leqq 1$ for every natural number n. If x and y commute, we have $(x+y)^r = \sum C^r{}_k x^k y^{r-k}$, so

$$[N(x+y)]^r = N((x+y)^r) = N(\sum C^r{}_k x^k y^{r-k}) \leqq \sum N(C^r{}_k \cdot e) N(x)^k N(y)^{r-k}$$
$$\leqq \sum N(x)^k N(y)^{r-k} \leqq (r+1)[\max(N(x), N(y))]^r.$$

By extracting r-th roots of the end terms and letting r become infinite, we obtain $N(x+y) \leqq \max(N(x), N(y))$.

THEOREM 3. *Let N be a PM pseudonorm of a commutative ring R, and let m be an integer with $m > 1$. Then N is non-archimedean if and only if $N(me) \leqq 1$.*

Proof. This follows obviously from Lemma 7.

COROLLARY. *Let N and N' be PM pseudonorms of a commutative ring R such that $N' \leqq N$. If N is non-archimedean, then N' is non-archimedean.*

LEMMA 8. *The join of a bounded family of non-archimedean pseudonorms on a ring R is a non-archimedean pseudonorm on R.*

COROLLARY. *A commutative PMN ring is non-archimedean if and only if every subordinate pseudo absolute value is non-archimedean.*

LEMMA 9. *If N is an absolute value for a division ring R and m is an integer with $m > 1$, then N is non-archimedean if and only if $N(me) \leqq 1$.*

Proof. If N is non-archimedean, we obviously have $N(me) \leqq 1$. Conversely, if $N(me) \leqq 1$ we apply Lemma 7, so that

$$N(x+y) = N(x)N(e+x^{-1}) \leqq N(x)\max(N(e), N(x^{-1}y)) = \max(N(x), N(y))$$

if $x \neq 0$, since e and $x^{-1}y$ commute.

THEOREM 4. *If N is a stable PM norm for a division ring R and m is an integer with $m > 1$, then N is non-archimedean if and only if $N(me) \leqq 1$.*

Proof. If $N(me) \leqq 1$, then $N'(me) \leqq 1$ for any absolute value N' subordinate to N, so N' is non-archimedean by Lemma 9. But N is the join of all absolute values subordinate to it, according to Corollary 2 of Theorem 2, and these are all non-archimedean; so N is non-archimedean by Lemma 8. The remainder of the theorem is obvious.

COROLLARY 1. *Exery subordinate stable PM norm of non-archimedean PMN division ring is non-archimedean.*

COROLLARY 2. *If $\{N_\lambda \mid \lambda \in \Lambda\}$ is a bounded family of stable PM norms of a division ring R, then $\bigvee\{N_\lambda \mid \lambda \in \Lambda\}$ is non-archimedean on R if and only if N_λ is non-archimedean for each λ in Λ.*

COROLLARY 3. *A stable PM norm N on a division ring is non-archimedean if and only if every absolute value subordinate to N is non-archimedean.*

Corollary 2 of Theorem 4, Lemma 8 and its corollary indicate that if a stable *PM* norm of a commutative ring or a division ring is expressed as the join of a family of pseudo absolute values, then the norm is non-archimedean if and only if every member of the family is non-archimedean. For an arbitrary ring, it may also be shown that a non-archimedean stable *PM* pseudonorm may always be expressed as the join of a bounded family of non-archimedean pseudo absolute values. (This is done by observing that the induced pseudonorms of a non-archimedean stable *PM* pseudonorm are also non-

archimedean; the proof of the Fundamental Theorem is then paralleled, but with the sets $[R;N;c]$ now restricted to their non-archimedean members.)

If an archimedean stable PM pseudonorm is expressed as the join of a family of pseudo absolute values, Lemma 8 implies that they are not all non-archimedean. There are then two possibilities: either all of these pseudo absolute values are archimedean, or at least one is non-archimedean. An archimedean stable PM pseudonorm will be called *strictly archimedean* if it may be expressed in at least one way as the join of archimedean pseudo absolute values; otherwise it will be called *partially archimedean.*

THEOREM 5. *If N is a partially archimedean stable PM pseudonorm of a ring R, then there exist non-zero stable PM pseudonorms N' and N'' such that N' is strictly archimedean on R, N'' is non-archimedean on R, and $N = N' \vee N''$.*

Proof. N is expressed as the join of a family of pseudo absolute values on R. Then N' and N'' are defined as the join of the archimedean members of the family, and the join of the non-archimedean members of the family, respectively.

The set of stable PM pseudonorms on a ring is thus divided into three classes: the non-archimedean, the strictly archimedean, and the partially archimedean. The partially archimedean pseudonorms resemble the strictly archimedean ones in some respects, and resemble the non-archimedean pseudonorms in other respects; Theorem 5 suggests this situation since it shows that a partially archimedean pseudonorm is obtained by joining a strictly archimedean and a non-archimedean pseudonorm. In the theorem which follows, the resemblance of partially archimedean pseudonorms to the non-archimedean ones is in evidence. The corollaries and the following theorem then give sufficient conditions for PMN division rings to be strictly archimedean.

LEMMA 10. *If a metric ring R admits a non-archimedean subordinate norm N', then R is totally disconnected.*

Proof. N' is continuous by Lemma 5, so for any positive ϵ the set $\mathfrak{G}(\epsilon) = \{x \mid N'(x) < \epsilon\}$ is an open set of R. Since N' is non-archimedean, it follows easily that $\mathfrak{G}(\epsilon)$ is an additive group for each positive ϵ, so $\mathfrak{G}(\epsilon)$ is open and closed in R. The intersection of all the $\mathfrak{G}(\epsilon)$ as ϵ ranges over the positive numbers is clearly 0. Since 0 is the intersection of a family of sets which are both open and closed, the connected component of 0 consists only of 0. This shows that R is totally disconnected.

THEOREM 6. *Let R be a non-archimedean metric ring or a partially archimedean simple PMN ring. Then R is totally disconnected.*

Proof. If R is a non-archimedean metric ring with norm N, let $N' = N$ and apply Lemma 10. If R is a partially archimedean simple *PMN* ring with norm N, then Theorem 5 states that there is a non-zero non-archimedean pseudonorm N'' subordinate to N. Since R is simple, N'' is a norm, according to Theorem 2. The application of Lemma 10 to N'' shows that R is totally disconnected in this case also.

COROLLARY 1. *If a simple PMN ring R is not totally disconnected, then R is strictly archimedean.*

COROLLARY 2. *Every connected PMN division ring is strictly archimedean.*

THEOREM 7. *Let R be a PMN division ring with norm N, and let m be an integer with $m > 1$. If $me \neq 0$ and $N((me)^{-1}) < 1$, then R is strictly archimedean.*

Proof. Since $N((me)^{-1}) < 1$, we have $N'((me)^{-1}) < 1$ for N' a component of N. But $N'(me)N'((me)^{-1}) = 1$, so $N'(me) > 1$, and Theorem 4 implies that N' is archimedean. Thus, all components of N are archimedean, and N is the join of its components, by Corollary 1 of Theorem 2, so that N is strictly archimedean.

6. Embedding theorems for archimedean division rings. A. Ostrowski showed in [4] that the archimedean fields with absolute value were algebraically and topologically isomorphic to subfields of the complex field. In this section, we shall generalize this result by showing that the archimedean *PMN* division rings are alegbraically isomorphic to division subrings of the division ring $\mathfrak{Q}$ of all real quaternions. In particular, the non-discrete, strictly archimedean *PMN* division rings will be shown to be algebraically and topologically isomorphic to division subrings of $\mathfrak{Q}$.

LEMMA 11. *If R is an archimedean division ring with absolute value, then R is algebraically and topologically isomorphic to a division subring of $\mathfrak{Q}$, and the norm in R corresponds to the ρ-th power of A, for some ρ such that $0 < \rho \leqq 1$.*

Proof. This follows from Theorem 11 of [2] and the statement following the corollary of that theorem.

THEOREM 8. *If R is an archimedean PMN division ring, then R is algebraically isomorphic to a division subring of $\mathfrak{Q}$.*

Proof. If N is the norm of R, then Corollary 1 of Theorem 2 states that N is the join of its components, which are all absolute values. Then Lemma 8 implies that the components are not all non-archimedean, for otherwise N would be non-archimedean. Let N' be an archimedean component of N, and apply Lemma 11 to R with N' taken as its norm. Then R is algebraically isomorphic to a division subring of $\mathfrak{Q}$.

LEMMA 12. *Let N be a non-zero pseudonorm of a ring R, and let Γ be a non-empty, bounded set of non-negative numbers such that N^γ is a pseudonorm for each γ in Γ. If $\alpha = \inf \Gamma$ and $\beta = \sup \Gamma$, then $\bigvee \{N^\gamma \mid \gamma \in \Gamma\} = N^\alpha \vee N^\beta$.*

Proof. The proof is elementary.

LEMMA 13. *Let R be a division subring of $\mathfrak{Q}$, and let N be a strictly archimedean stable PM norm on R. Then there exist real numbers α and β, with $0 < \alpha \leqq \beta \leqq 1$ or $0 = \alpha < \beta \leqq 1$, such that $N = A^\alpha \vee A^\beta$.*

Proof. Any archimedean absolute value on R must be of the form A^γ for some γ such that $0 < \gamma \leqq 1$. This follows from Lemma 11. But a strictly archimedean stable PM norm on a division ring is the join of a non-empty family of archimedean absolute values, so $N = \bigvee \{A^\gamma \mid \gamma \in \Gamma\}$, where Γ is a non-empty subset of $]0, 1]$. Then Lemma 12 implies that $N = A^\alpha \vee A^\beta$, with $\alpha = \inf \Gamma$ and $\beta = \sup \Gamma$. Clearly, $0 \leqq \alpha \leqq \beta \leqq 1$, and β is positive. This completes the proof.

THEOREM 9. *Let R be a strictly archimedean, non-discrete PMN division ring. Then R is algebraically and topologically isomorphic to a division subring of Q.*

Proof. By Theorem 8, R is algebraically isomorphic to a division subring of $\mathfrak{Q}$. If we choose a specific algebraic isomorphism, we may consider R as a subset of $\mathfrak{Q}$. Lemma 13 shows that the norm of R is of the form $A^\alpha \vee A^\beta$, with $0 \leqq \alpha \leqq \beta \leqq 1$. Since R is not discrete, we must have α positive, for the norm $A^0 \vee A^\beta = T \vee A^\beta$ is clearly discrete. Then $A^\alpha \vee A^\beta$, A^α, and A give the same topology on R, and so the topology given to R by its norm is the same as the relative topology of R considered as a subset of $\mathfrak{Q}$. Thus, the algebraic isomorphism of R into $\mathfrak{Q}$ is also a topological isomorphism (homeomorphism).

COROLLARY 1. *Let R be a strictly archimedean, non-discrete, complete PMN division ring. Then R is algebraically and topologically isomorphic to the real field $\mathfrak{R}$, the complex field $\mathfrak{C}$, or $\mathfrak{Q}$.*

COROLLARY 2. *Let R be a connected PMN division ring. Then R is algebraically and topologically isomorphic to a division subring of $\mathfrak{Q}$.*

COROLLARY 3. *Let R be a connected, complete PMN division ring. Then R is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

7. *PM* pseudonorms for special cases. In this section, we shall describe all non-zero *PM* pseudonorms of the rational field or of the ring of integers.

It is well known (see, for instance, chapter X of [5]) that the only absolute values of either of these rings are T, A^γ with $0 < \gamma \leqq 1$, and $B_p{}^\gamma$ with p a prime number and $\gamma > 0$. Also, it is easily proved that the only other non-zero pseudo absolute values of the ring of integers are of the type T_p, with p prime, where T_p is the characteristic function of the set of integers not divisible by p. Theorem 1 indicates that all non-zero *PM* pseudonorms of the ring of integers may be obtained by taking joins of pseudo absolute values of these four types, while Theorems 1 and 2 indicate that a *PM* norm of the rational field is the join of absolute values of the first three types. By the use of Lemma 12, these expressions may be simplified so that at most two powers of a given pseudo absolute value appear. Further simplifications may be made by the use of simple manipulations. It will suffice to describe the strictly archimedean and the non-zero non-archimedean cases, for the partially archimedean case is then easily described by using Theorem 5.

THEOREM 10. *Let N be a strictly archimedean PM norm of the rational field. Then either $N = A^\alpha \vee A^\beta$, with $0 < \alpha \leqq \beta \leqq 1$, or $N = T \vee A^\beta$, with $0 < \beta \leqq 1$.*

THEOREM 11. *Let N be a non-archimedean PM norm of the rational field. Then $N = T$, or $N = \vee \{B_p{}^{\alpha(p)} \vee B_p{}^{\beta(p)} \mid p \in \mathfrak{P}\}$ with $\mathfrak{P}$ a finite non-empty set of primes and $0 < \alpha(p) \leqq \beta(p)$ for all p in $\mathfrak{P}$, or*

$$N = T \vee [\vee \{B_p{}^{\beta(p)} \mid p \in \mathfrak{P}\}]$$

with $\mathfrak{P}$ a finite non-empty set of primes and $0 < \beta(p)$ for all p in $\mathfrak{P}$, or $N = \vee \{B_p{}^{\beta(p)} \mid p \in \mathfrak{P}\}$ with $\mathfrak{P}$ an infinite set of primes and $0 < \beta(p)$ for all p in $\mathfrak{P}$.

THEOREM 12. *Let N be a strictly archimedean PM pseudonorm on the ring of integers. Then $N = A^{\beta}$, with $0 < \beta \leqq 1$.*

THEOREM 13. *Let N be a non-zero non-archimedean PM pseudonorm of the ring of integers. Then $N = T$, or*

$$N = [\vee \{B_p{}^{\alpha(p)} \mid p \in \mathfrak{P}\}] \vee [\vee \{T_p \mid p \in \mathfrak{P}'\}],$$

where $\mathfrak{P}$ and $\mathfrak{P}'$ are disjoint finite sets of primes, not both empty, and $0 < \alpha(p)$ for all p in $\mathfrak{P}$.

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THE SUSPENSION OF A LOOP SPACE.*

By W. D. Barcus and J.-P. Meyer.[1]

Part I. The Homology Suspension.

1. Introduction. If X is a topological space, let SX denote the suspension of X, i.e., the join of X with a pair of points p_+, p_-, and let $\Omega(X, x_0)$ denote the space of loops on X at x_0. If Y is another space, let $M(Y, X)$ denote the set of all (continuous) maps $Y \to X$, and similarly for pairs and triples. There is a well-known 1-1 correspondence

$$\psi\colon M((SY, p_+, p_-), (X, x_0, x_0)) \to M(Y, \Omega(X, x_0)).$$

Let $i\colon SX \to \mathcal{S}X$ be the identification map, where $\mathcal{S}X$ is the reduced suspension. G. W. Whitehead [17] studied the homotopy suspension $E\colon \pi_n(X) \to \pi_{n+1}(SX)$ by using the map $\psi(i)\colon X \to \Omega\mathcal{S}X$. We consider a dual situation: abbreviate $\Omega(X, x_0)$ by Ω, and let $j\colon \Omega \to \Omega$ be the identity. Then the map $\psi^{-1}(j)\colon S\Omega \to X$ induces homomorphisms of the homology groups which are closely related to the homology suspension $\sigma\colon H_n(\Omega) \to H_{n+1}(X)$. It is convenient to convert $\psi^{-1}(j)$ into an equivalent fibre map. The fibre is of the homotopy type of the join $\Omega * \Omega$, and the Serre homology sequence of the fibering is essentially the same as G. W. Whitehead's sequence [18] involving σ, but contains an extra term. This gives an alternative proof of Whitehead's main result, and also allows us to extend several of his corollaries by one dimension: e.g. a cohomology operation of type $(n, q; \pi, G)$, $q \leqq 3n$, is additive if and only if it is a suspension. As a further application, in Part II we apply the above fibering to the problem of calculating the Postnikov invariants of the suspension of an Eilenberg-MacLane space $K(\pi, n)$.

2. Preliminaries. The homology theory used in this paper is the cubical singular theory of Serre [14] as modified by Bott and Samelson [3].

The Suspension. Let I be the unit interval. The suspension of X, SX,

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may be defined as the space obtained from $X \times I$ by the identifications $(x,0) = (x',0)$ and $(x,1) = (x',1)$ for all $x, x' \in X$. Let $\eta: X \times I \to SX$ be the identification map; we shall write the point $\eta(x,t)$ as $[x,t]$ and set $p_- = [x,0]$, $p_+ = [x,1]$. The point p_- will usually be taken as a base-point. SX contains subspaces X_+, X, X_- which are, respectively, $\eta(X \times \{t \mid \frac{1}{2} \leqq t \leqq 1\})$, $\eta(X \times \frac{1}{2})$, and $\eta(X \times \{t \mid 0 \leqq t \leqq \frac{1}{2}\})$. The *suspension* $E: H_n(X) \to H_{n+1}(SX)$, not to be confused with the homology suspension (defined below), is the composition $j_*^{-1} i_* \partial^{-1}$:

$$H_n(X) \xleftarrow{\partial} H_{n+1}(X_+, X) \xrightarrow{i_*} H_{n+1}(SX, X_-) \xleftarrow{j_*} H_{n+1}(SX) \qquad (n > 0).$$

We give a direct description of E. For a singular cube $f \in C_n(X)$, define $Sf \in C_{n+1}(SX)$ by

$$Sf(t_1, \cdots, t_{n+1}) = [f(t_1, \cdots, t_n), t_{n+1}]. \tag{2.1}$$

S commutes with the boundary operator for $n > 0$, and hence defines a homomorphism of the homology groups, still denoted by S.

(2.2) PROPOSITION. $E = (-1)^{n+1} S: H_n(X) \to H_{n+1}(SX) \quad (n > 0)$.

The proof is elementary, and is omitted.

The Homology Suspension. Let $(X; A, B)$ be a triad. Following Serre [14], we denote by $E_{A,B}(X)$ the space of (continuous) paths in X, starting in A and ending in B. If $x_0 \in X$, we abbreviate $E_{x_0,x_0}(X)$ by $\Omega(X, x_0)$. The map $p: E_{x_0,X}(X) \to X$ which assigns to each path its endpoint is a fibre map, the total space is contractible, and $p^{-1}(A) = E_{x_0,A}(X)$. The homomorphism

$$H_n(E_{x_0,A}(X)) \xrightarrow{\partial^{-1}} H_{n+1}(E_{x_0,X}(X), E_{x_0,A}(X)) \xrightarrow{p_*} H_{n+1}(X, A) \qquad (n > 0)$$

is well-defined, and is denoted by σ_A; σ_{x_0} is abbreviated by σ, and is called the *homology suspension.* Similarly, the map $p^\#: E_{X,x_0}(X) \to X$ which assigns to each path its initial point is a fibre map; the homomorphism corresponding to σ_A will be denoted $\sigma_A{}^\#: H_n(E_{A,x_0}(X)) \to H_{n+1}(X, A)$. By a slight modification of the argument on p. 482 of [14], one verifies that $(-1)^n \sigma_A$ is induced by a homomorphism $s: C_n(E_{x_0,A}(X)) \to C_{n+1}(X, A)$ defined on singular cubes by

$$sf(t_1, \cdots, t_{n+1}) = f(t_1, \cdots, t_n)(t_{n+1}). \tag{2.3}$$

(2.4) PROPOSITION. *If (X, A) is q-connected and X is r-connected, then σ_A is an isomorphism for $0 < n < q + r$. A similar result hold for $\sigma_A{}^\#$.*

Proof. The fibre is $\Omega(X, x_0)$, which is $(r-1)$-connected. The result then follows from Theorem 1B, p. 268 of [15].

The Fundamental Map. For the remainder of Part I, X will denote a fixed topological space, $x_0 \in X$ a base-point possessing a contractible neighborhood in X, and $\Omega(X, x_0)$ will be abbreviated by Ω.

The *fundamental map* $k: S\Omega \to X$ is defined by $k[\omega, t] = \omega(t)$, $\omega \in \Omega$.

(2.5) PROPOSITION. $k_* E = -\sigma: H_n(\Omega) \to H_{n+1}(X)$.

Proof. Let $f \in C_n(\Omega)$ be a singular cube. Then

$$k(Sf)(t_1, \cdots, t_{n+1}) = k[f(t_1, \cdots, t_n), t_{n+1}] = f(t_1, \cdots, t_n)(t_{n+1})$$
$$= sf(t_1, \cdots, t_{n+1}).$$

Hence $(-1)^{n+1} k_* E = k_* S = (-1)^n \sigma$.

Homotopy Properties. There are well-known analogues of E and σ in homotopy; we recall briefly their definitions.

Let $x_0 \in \Omega$ denote the constant loop at $x_0 \in X$. Let $\alpha \in \pi_n(\Omega, x_0)$ be represented by $f: (I^n, \dot{I}^n) \to (\Omega, x_0)$; then $(-1)^n \sigma(\alpha) \in \pi_{n+1}(X, x_0)$ is represented by $f': (I^{n+1}, \dot{I}^{n+1}) \to (X, x_0)$, where

$$f'(t_1, \cdots, t_{n+1}) = f(t_1, \cdots, t_n)(t_{n+1}).$$

By exactness, the injection $i_*: \pi_{n+1}(SX, p_-) \to \pi_{n+1}(SX, Sx_0)$ is an isomorphism for all n. If $\beta \in \pi_n(X, x_0)$ is represented by $g: (I^n, \dot{I}^n) \to (X, x_0)$, then $(-1)^{n+1} E\beta = {i_*}^{-1}\gamma$, where γ is the class of $g': (I^{n+1}, \dot{I}^{n+1}) \to (SX, Sx_0)$, defined by

$$g'(t_1, \cdots, t_{n+1}) = [g(t_1, \cdots, t_n), t_{n+1}].$$

(2.6) PROPOSITION. $k_* E = -\sigma: \pi_n(\Omega, x_0) \to \pi_{n+1}(X, x_0)$.

The proof is similar to that in homology.

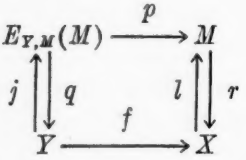
From (2.6) and the fact that σ is an isomorphism, we conclude

(2.7) COROLLARY. $k_*: \pi_{n+1}(S\Omega, p_-) \to \pi_{n+1}(X, x_0)$ *is an epimorphism, and* $E: \pi_n(\Omega, x_0) \to \pi_{n+1}(S\Omega, p_-)$ *is a monomorphism.*

Converting a Map into a Fibre Map. The following construction is due to Cartan and Serre [5]. Let $f: Y \to X$ be a map, and let M be the mapping cylinder of f: i.e. the space obtained from the disjoint union $Y \times I \cup X$ by identifying $(y, 1) = f(y) \in X$ for all $y \in Y$. Y and X are embedded in M as the images of the sets $Y \times 0$ and X respectively. The image in M of $(y, t) \in Y \times I$ will again be denoted by (y, t). Define $p: E_{Y,M}(M) \to M$ by

$p(\tau) = \tau(1)$; then p is a fibre map, and is equivalent to f in the following sense. Consider the diagram

(2.8)



where $j(y)$ is the constant path at y; $q(\tau) = \tau(0)$; l is the inclusion; and r is the retraction $r(y,t) = f(y)$, $r(x) = x$. Then qj and rl are equal to the appropriate identity maps, jq and lr are homotopic to the identities, and $rpj = f$.

If $f: Y \to X$ is an inclusion, we need not introduce the mapping cylinder: in this case we replace M by X and $l = r =$ identity.

3. The fibre space. In this and the next section, we study the fundamental map $k: S\Omega \to X$ by converting it into the fibre map $p: E_{S\Omega,M}(M) \to M$ described in § 2. As a base-point in M, we take $x_0 \in X$; then the fibre over x_0 is $F = E_{S\Omega,x_0}(M)$.

The join $\Omega * \Omega$ is defined to be the space obtained from $\Omega \times I \times \Omega$ by identifying $(\mu, 0, \nu) = (\mu, 0, \nu')$ and $(\mu, 1, \nu) = (\mu', 1, \nu)$ for all $\mu, \mu', \nu, \nu' \in \Omega$. The point of $\Omega * \Omega$ which is the image of (μ, t, ν) will again be denoted (μ, t, ν). We define a function $h: \Omega * \Omega \to F$ as follows. If $y \in S\Omega$, let $\lambda(y)$ be the path in M from y to $k(y) \in X$ given by $\lambda(y)(s) = (y, s)$. If $\mu \in \Omega$ and $0 \leqq a \leqq b \leqq 1$, we define the path $\mu_{a,b}$ from $\mu(a)$ to $\mu(b)$ by $\mu_{a,b}(s) = \mu((b-a)s + a)$. The inverse of $\mu_{a,b}$ will be denoted by $\mu_{a,b}^{-1}$. Then if $(\mu, t, \nu) \in \Omega * \Omega$,

$$h(\mu, t, \nu) = \begin{cases} \lambda[\mu\nu, t]\mu_{2t,1} & \text{if } 0 \leqq t \leqq \frac{1}{2} \\ \lambda[\mu\nu, t]\nu_{0,2t-1}^{-1} & \text{if } \frac{1}{2} \leqq t \leqq 1. \end{cases} \tag{3.1}$$

(3.2) PROPOSITION. *The function h is well-defined and continuous. Furthermore, h is a homotopy equivalence, so that F is of the homotopy type of $\Omega * \Omega$.*

The proof is given in the appendix.

4. The homology sequence. Let $p: E \to B$ be a fibre space in which B is q-connected, $q \geqq 1$, and the fibre F is r-connected. Then by Corollary 1, p. 469 of [14], $p_*: H_i(E, F) \to H_i(B, b_0)$ is an isomorphism for $i \leqq q + r + 1$, and an epimorphism for $i = q + r + 2$, provided the coefficients are in a

principal ideal ring Γ. According to Lemma 5.2 of [18], the same result therefore holds with coefficients in any Γ-module A. From the homology sequence of the pair (E, F) with coefficients in A, we thus derive the exact Serre sequence of the fibering ([14], p. 468) with coefficients in A:

$$H_{q+r+1}(F) \xrightarrow{i_*} H_{q+r+1}(E) \xrightarrow{p_*} H_{q+r+1}(B) \xrightarrow{\tau} H_{q+r}(F) \to \cdots.$$

Here i_* denotes the injection and τ the transgression.

Throughout the remainder of this section, all homology groups are to have coefficients in a fixed Abelian group G.

Let X be n-connected, $n \geqq 1$. Then $\Omega * \Omega$ is $2n$-connected, and the Serre sequence of the fibering of § 3 is (using the above remarks with Γ the ring of integers, $A = G$)

$$(4.1) \qquad H_{3n+1}(F) \xrightarrow{i_*} H_{3n+1}(E_{S\Omega,M}(M)) \xrightarrow{p_*} H_{3n+1}(M) \xrightarrow{\tau} \cdots.$$

In this section, we shall give alternative expressions for the homomorphisms in (4.1).

Recall that Lemma (2.5) of [18] states that there is a unique homomorphism ρ such that the following diagram is commutative for $i \geqq 1$, where $\Omega \vee \Omega = \Omega \times x_0 \cup x_0 \times \Omega$, j' is the inclusion, m is the multiplication in Ω, $p_1, p_2 : \Omega \times \Omega \to \Omega$ are the projections, and $\xi = m_* - p_{1*} - p_{2*}$:

$$\begin{array}{ccc} H_i(\Omega \times \Omega) & \xrightarrow{j'_*} & H_i(\Omega \times \Omega, \Omega \vee \Omega) \\ & {\scriptstyle \xi} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & H_i(\Omega) & \end{array}$$

(4.2) Proposition. *There exists a commutative diagram for $i > 1$, in which θ_1, θ_2 and r_* are isomorphisms:*

$$(4.3) \qquad \begin{array}{ccccc} H_i(F) & \xrightarrow{i_*} & H_i(E_{S\Omega,M}(M)) & \xrightarrow{p_*} & H_i(M) \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle r_*} \\ H_{i-1}(\Omega \times \Omega, \Omega \vee \Omega) & \xrightarrow{\rho} & H_{i-1}(\Omega) & \xrightarrow{\sigma} & H_i(X) \end{array}$$

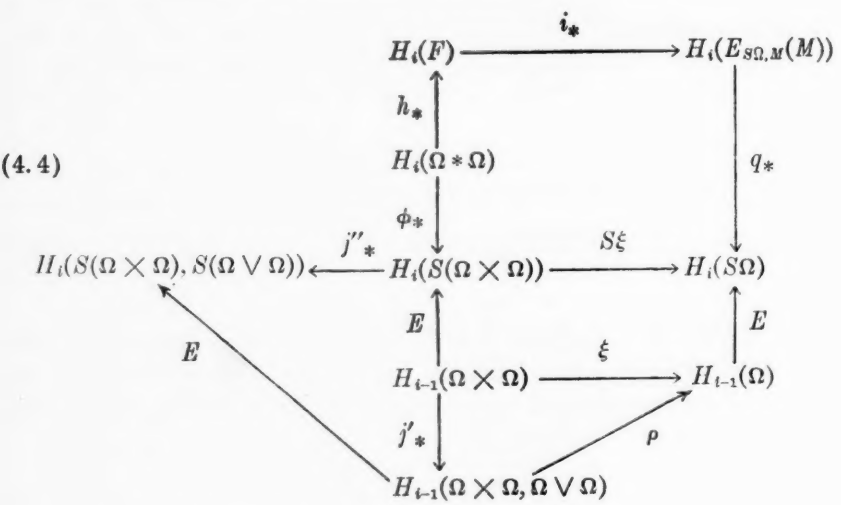
Proof. Consider the second square, in which $\theta_2 = -E^{-1}q_*$; q and r are defined in diagram (2.8). According to the remarks following (2.8), with $f = k$, we have $k_*q_* = r_*p_*$; hence, using (2.5),

$$\sigma\theta_2 = -\sigma E^{-1}q_* = k_*q_* = r_*p_*.$$

Since q and r are homotopy equivalences, q_* and r_* are isomorphisms. E is an isomorphism, and hence so is θ_2.

To prove commutativity in the first square, we consider the following diagram, in which $\phi: \Omega * \Omega \to S(\Omega \times \Omega)$ is defined by $\phi(\mu, t, \nu) = [(\mu, \nu), t]$, Sm is the suspension of m, and $S\xi = (Sm)_* - (Sp_1)_* - (Sp_2)_*$:

(4.4)



Now $qih(\mu, t, \nu) = \begin{cases} q(\lambda[\mu\nu, t]\mu_{2t,1}) & \text{if } 0 \leqq t \leqq \frac{1}{2} \\ q(\lambda[\mu\nu, t]\nu_{0,2t-1^{-1}}) & \text{if } \frac{1}{2} \leqq t \leqq 1 \end{cases}$

$= [\mu\nu, t]$;

also, $Sm\phi(\mu, t, \nu) = [\mu\nu, t]$;

hence $q_*i_*h_* = (Sm)_*\phi_* = S\xi\phi_*$, since $(Sp_1)_*\phi_* = (Sp_2)_*\phi_* = 0$, and the top rectangle of (4.4) is commutative. The remainder of the diagram is obviously commutative.

One deduces immediately from the proof of Lemma (7.2) in [19] that $j''_*\phi_*$ is an isomorphism; hence so are $E^{-1}j''_*\phi_* = j'_*E^{-1}\phi_*$ and $\theta_1 = -j'_*E^{-1}\phi_*h_*^{-1}$. Since $\theta_2 = -E^{-1}q_*$, commutativity of (4.4) implies that $\rho\theta_1 = \theta_2 i_*$. This completes the proof of (4.2).

Henceforth, we shall set $S\Omega \vee S\Omega = S\Omega \times p_- \cup p_+ \times S\Omega \subset S\Omega \times S\Omega$, and $M \vee M = M \times p_- \cup p_+ \times M$. It is well-known that the diagonal map $d_0 : S\Omega \to S\Omega \times S\Omega$ is homotopic to $i_1 + i_2$, where

$$(i_1 + i_2)[\omega, t] = \begin{cases} [\omega, 2t] \times p_- & \text{if } 0 \leqq t \leqq \frac{1}{2} \\ p_+ \times [\omega, 2t-1] & \text{if } \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

A homotopy from d_0 to $i_1 + i_2$ is given by

$$\begin{aligned} d_u[\omega, t] &= [\omega, t + ut] \times [\omega, t - ut] \quad \text{if } 0 \leqq t \leqq \tfrac{1}{2} \\ &= [\omega, t + u(1-t)] \times [\omega, (1+u)t - u] \quad \text{if } \tfrac{1}{2} \leqq t \leqq 1. \end{aligned}$$

Let $\bar{M}$ be the mapping cylinder of $k \times k : S\Omega \times S\Omega \to X \times X$, and $\zeta : \bar{M} \subset M \times M$ the embedding defined by

$$\zeta([\omega, t], [\omega', t'], u) = (([\omega, t], u), ([\omega', t'], u)).$$

We define a map $d' : M \to M \times M$ by

$$d'([\omega, t], u) = \zeta((d_{1-u}[\omega, t]), u). \tag{4.5}$$

Let $d : M \to M \times M$ be the diagonal map. It is easily verified that

$$d' \mid S\Omega = i_1 + i_2, \quad d' \mid X = d \mid X, \text{ and } d' \cong d \operatorname{rel} X;$$

in particular, as a map of pairs, $d' : (M, S\Omega) \to (M \times M, M \vee M)$.

Let $(M \vee M)^*$ be the cone on $M \vee M$, let M' be the union $M \times M \cup (M \vee M)^*$ such that $(M \times M) \cap (M \vee M)^* = M \vee M$, and let

$$a : (M \times M, M \vee M) \to (M', (M \vee M)^*)$$

be the inclusion. Then d' and a induce maps of the path spaces

$$\begin{array}{ccccc} E_{S\Omega, M}(M) & \to & E_{M \vee M, M \times M}(M \times M) & \to & E_{(M \vee M)^*, M'}(M') \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} \\ M & \xrightarrow{d'} & M \times M & \xrightarrow{a} & M' \end{array}$$

which, restricted to the fibres, yield maps (abbreviating $x_0 \times x_0$ by x_0)

$$F = E_{S\Omega, x_0}(M) \xrightarrow{d_1'} E_{M \vee M, x_0}(M \times M) \xrightarrow{a_1} E_{(M \vee M)^*, x_0}(M').$$

Since the transgression and homology suspension are natural, the above maps of fibre spaces, and the corresponding ones for $p^{\#}$, $p_2{}^{\#}$, $p_3{}^{\#}$ (cf. § 2), yield a commutative diagram for $i \geqq 2$

$$\begin{array}{ccccccc}
H_i(X) & \xleftarrow{r_*} & H_i(M) & \xrightarrow{\tau} & H_{i-1}(E_{S\Omega,x_0}(M)) & \xrightarrow{\sigma^{\#}} & H_i(M, S\Omega) \\
\downarrow{\scriptstyle d_*} & & \downarrow{\scriptstyle d'_*} & & \downarrow{\scriptstyle d'_{1*}} & & \downarrow{\scriptstyle d'_*} \\
H_i(X \times X) & \xleftarrow{(r \times r)_*} & H_i(M \times M) & \xrightarrow{\tau_2} & H_{i-1}(E_{M \vee M, x_0}(M \times M)) & \xrightarrow{\sigma_2{}^{\#}} & H_i(M \times M, M \vee M) \\
\downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle a_{1*}} & & \downarrow{\scriptstyle a_*} \\
H_i(X') & \xleftarrow{r'_*} & H_i(M') & \xrightarrow{\tau_3} & H_{i-1}(E_{(M \vee M)^*, x_0}(M')) & \xrightarrow{\sigma_3{}^{\#}} & H_i(M', (M \vee M)^*).
\end{array}$$

Here $X \vee X = X \times x_0 \cup x_0 \times X$, and we have used the same notation as for M: $(X \vee X)^*$ is the cone on $X \vee X$, $X' = X \times X \cup (X \vee X)^*$ and d and a are the diagonal and inclusion maps. r' is induced by r; since r is a homotopy equivalence, r'_* is an isomorphism for all i. τ, τ_2 and τ_3 are transgressions: τ and τ_2 are defined for $i \leqq 3n+1$, τ_3 for $i \leqq 4n+2$. $\sigma^{\#}$, $\sigma_2{}^{\#}$ and $\sigma_3{}^{\#}$ are the generalized suspensions of § 2; by Proposition (2.4), $\sigma^{\#}$ and $\sigma_2{}^{\#}$ are isomorphisms for $i-1 < 3n+1$, $\sigma_3{}^{\#}$ for $i-1 < 4n+2$. Hence, if we define $\theta'_3 : H_{i-1}(E_{S\Omega,x_0}(M)) \to H_i(X')$ by $\theta'_3 = r'_* \tau_3{}^{-1} \sigma_3{}^{\#-1} a_* d'_* \sigma^*$ for $i \leqq 4n+2$, then we have a commutative diagram

$$(4.7) \qquad \begin{array}{ccc}
H_i(M) & \xrightarrow{\tau} & H_{i-1}(E_{S\Omega,x_0}(M)) \\
\downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle \theta'_3} \\
H_i(X) & \xrightarrow{a_* d_*} & H_i(X').
\end{array}$$

(4.8) LEMMA. *Let X be n-connected, $n \geqq 2$. Then $d'_* : H_i(M, S\Omega) \to H_i(M \times M, M \vee M)$ is an isomorphism for $i \leqq 3n$, and an epimorphism for $i = 3n+1$.*

The proof is given in the appendix.

Using the following commutative diagram of injections, we can state (4.7) in a more convenient form:

$$\begin{array}{ccc}
H_i(X \times X) & \xrightarrow{a_*} & H_i(X') \\
\downarrow{\scriptstyle b_*} & & \downarrow{\scriptstyle b'_*} \\
H_i(X \times X, X \vee X) & \xrightarrow{a'_*} & H_i(X', (X \vee X)^*).
\end{array}$$

a'_* is an isomorphism by excision, and b'_* is an isomorphism for $i > 0$; hence we may define $\theta_3 : H_{i-1}(E_{S\Omega,x_0}(M)) \to H_i(X \times X, X \vee X)$ by $\theta_3 = a'^{-1}_* b'_* \theta'_3$. Abbreviating $b_* d_*$ by d_*, and reverting to the notation $F = E_{S\Omega,x_0}(M)$, we have

(4.9) PROPOSITION. *There exists a commutative diagram for* $1 < i \leqq 4n+2$

$$\begin{array}{ccc} H_i(M) & \xrightarrow{\tau} & H_{i-1}(F) \\ {\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle \theta_3} \\ H_i(X) & \xrightarrow{d_*} & H_i(X \times X, X \vee X) \end{array}$$

such that if $n \geqq 2$, *then* θ_3 *is an isomorphism for* $i < 3n+1$ *and an epimorphism for* $i = 3n+1$.

Results which are dual to the above hold for cohomology: Firstly, the dual Serre sequence is

$$(4.10)\quad H^{3n+1}(F) \xleftarrow{i^*} H^{3n+1}(E_{S\Omega,M}(M)) \xleftarrow{p^*} H^{3n+1}(M) \xleftarrow{\tau^*} H^{3n}(F) \leftarrow \cdots.$$

Secondly, the following diagram is commutative for $i > 1$:

$$(4.11)\quad \begin{array}{ccccc} H^i(F) & \xleftarrow{\quad i^*\quad} & H^i(E_{S\Omega,M}(M)) & \xleftarrow{\quad p^*\quad} & H^i(M) \\ \uparrow{\scriptstyle \theta_1^*} & & \uparrow{\scriptstyle \theta_2^*} & & \uparrow{\scriptstyle r^*} \\ H^i(\Omega \times \Omega, \Omega \vee \Omega) & \xleftarrow{\rho^*} & H^{i-1}(\Omega) & \xleftarrow{\sigma^*} & H^i(X), \end{array}$$

where all the homomorphisms are the duals of those previously defined; and finally, there is a commutative diagram for $1 < i \leqq 4n+2$

$$(4.12)\quad \begin{array}{ccc} H^i(M) & \xleftarrow{\tau^*} & H^{i-1}(F) \\ {\scriptstyle r^*}\uparrow & & \uparrow{\scriptstyle \theta_3^*} \\ H^i(X) & \xleftarrow{d^*} & H^i(X \times X, X \vee X) \end{array}$$

such that if $n \geqq 2$, then θ_3^* is an isomorphism for $i < 3n+1$ and a monomorphism for $i = 3n+1$.

Hence, with the one exception that θ_3^* has only been proved a monomorphism in dimension $3n+1$, we have shown that, for $n \geqq 2$, the sequence

(4.10) is the same as that obtained by G. W. Whitehead in [18], but with the addition of a new term $H^{3n+1}(F) \approx H^{3n}(\Omega \times \Omega, \Omega \vee \Omega)$. The presence of the additional term allows us to extend Corollaries 6.2, 6.3 and 7.2 of [18] by one dimension; we state only the last of these.

(4.13) COROLLARY. *Let θ be a cohomology operation of type $(n, q; \pi, G)$. If the corresponding element $u(\theta) \in H^q(\pi, n: G)$ is a suspension element, then θ is additive. Conversely, if $q \leqq 3n$ and θ is additive, then $u(\theta)$ is a suspension element.*

It should be noted that Moore [12] has given an example of an operation θ, for which $q = 3n + 3$, which is additive but is not a suspension.

Proof of (4.13). By Lemma (7.1) of [18], θ is additive if and only if $u(\theta)$ is primitive (i.e. $\rho^* u(\theta) = 0$). In (4.11), $i^* p^* = 0$; but since θ_1^*, θ_2^* and r^* are isomorphisms and the diagram commutes, this implies that $\rho^* \sigma^* = 0$, which implies the first statement of the corollary. Since (4.10) is exact, the top line of (4.11) is exact for $i \leqq 3n + 1$, and hence so is the bottom line; but this implies the second statement of the corollary.

Appendix

(A.1) *Proof of Proposition* (3.2).[2]

Recalling that $F = E_{S\Omega, x_0}(M)$, we define $h' : F \to \Omega * \Omega$ as follows: let $\alpha \in F$, $[\omega, t] = \alpha(0)$, $\beta = r\alpha \in E_{X, x_0}(X)$; then

$$h'(\alpha) = (\omega_{0,t}\beta, t, \beta^{-1}\omega_{t,1}).$$

Both h and h' are readily seen to be well-defined and continuous.

We shall use the notation $\omega\{u, v\}$, $0 \leqq u \leqq v \leqq 1$, to indicate that the path ω is traversed in the interval $[u, v]$; e.g. $\omega\omega' = \omega\{0, \frac{1}{2}\}\omega'\{\frac{1}{2}, 1\}$. There are several ways to deform a path:

(1) by deforming its interval of definition, $\omega\{a, b\} \to \omega\{c, d\}$

(2) by a change in parametrization: e.g. $\omega_{t,t}\{a, b\}\omega'\{b, c\} \to \omega'\{a, c\}$

(3) by deforming a path of the form $\omega^{-1}\{a, b\}\omega\{b, c\}$ to the trivial path $\omega_{0,0}\{a, c\}$.

I) By using an appropriate sequence of these operations, one verifies that $h'h \cong G \cong H \cong 1$. where 1 is the identity map of $\Omega * \Omega$, and G, H are defined below: for $0 \leqq t \leqq \frac{1}{2}$, setting $\omega = \mu_{2t,1}{}^{-1}(\mu\nu)_{t,1}$,

[2] The authors' original proof was indirect; this proof was suggested by E. H. Brown.

$$G(\mu, t, \nu) = ((\mu\nu)_{0,t}\mu_{2t,1}, t, \omega)$$
$$= (\mu_{0,2t}\mu_{2t,1}, t, \omega)$$
$$H(\mu, t, \nu) = (\mu, t, \omega)$$
$$= (\mu, t, \mu_{2t,1}{}^{-1}\{0, \tfrac{1}{2}\}\mu_{2t,1}\{\tfrac{1}{2}, c\}\nu\{c, 1\}),$$

where $c = (3-4t)/(4-4t)$. For $\frac{1}{2} \leqq t \leqq 1$, setting $\tau = (\mu\nu)_{0,t}\nu_{0,2t-1}{}^{-1}$,

$$G(\mu, t, \nu) = (\tau, t, \nu_{0,2t-1}(\mu\nu)_{t,1})$$
$$= (\tau, t, \nu_{0,2t-1}\nu_{2t-1,1})$$
$$H(\mu, t, \nu) = (\tau, t, \nu)$$
$$= (\mu\{0, e\}\nu_{0,2t-1}\{e, \tfrac{1}{2}\}\nu_{0,2t-1}{}^{-1}\{\tfrac{1}{2}, 1\}, t, \nu),$$

where $e = 1/(4t)$.

II) The composition $hh'\colon F \to F$ is given by

$$hh'(\alpha) = \lambda[(\omega_{0,t}\beta)(\beta^{-1}\omega_{t,1}), t](\omega_{0,t}\beta)_{2t,1} \qquad \text{if } 0 \leqq t \leqq \tfrac{1}{2}$$
$$\lambda[(\omega_{0,t}\beta)(\beta^{-1}\omega_{t,1}), t](\beta^{-1}\omega_{t,1})_{0,2t-1}{}^{-1} \qquad \text{if } \tfrac{1}{2} \leqq t \leqq 1.$$

Let $T\colon F \to F$ be given by $T(\alpha) = \lambda[\omega, t]\beta$; then $T \cong 1'$, where $1'$ is the identity map of F. Using the above deformations of paths, we shall indicate a homotopy $hh' \cong T$ for $0 \leqq t \leqq \frac{1}{4}$; closely analogous homotopies can be given for the other three quarters of the unit t-interval. $hh' \cong H_1 \cong H' \cong T$, where H_1 and H' are defined for $0 \leqq t \leqq \frac{1}{4}$ as follows: Let H_s be the homotopy given by

$$H_s(\alpha) = \lambda[\omega_{0,t}\{0, a\}\beta\{a, \tfrac{1}{2}\}\beta^{-1}\{\tfrac{1}{2}, \tfrac{3}{4}\}\omega_{t,1}\{\tfrac{3}{4}, 1\}, t]\{0, \tfrac{1}{2}\}\omega_{b,t}\{\tfrac{1}{2}, d\}\beta\{d, 1\},$$

where $a = (4ts - s + 1)/4$, $b = 4t^2/(4ts - s + 1)$, and $d = (3-8t)/(4-8t)$. Then $H_0(\alpha) = hh'(\alpha)$, and

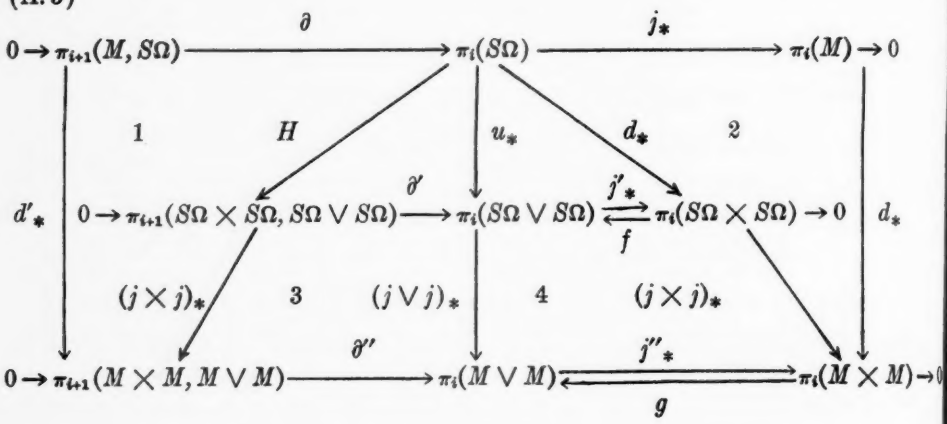
$$H_1(\alpha) = \lambda[\omega_{0,t}\{0, t\}\beta\{t, \tfrac{1}{2}\}\beta^{-1}\{\tfrac{1}{2}, \tfrac{3}{4}\}\omega_{t,1}\{\tfrac{3}{4}, 1\}, t]\{0, \tfrac{1}{2}\}\omega_{t,t}\{\tfrac{1}{2}, d\}\beta\{d, 1\}$$
$$H'(\alpha) = \lambda[\omega_{0,t}\{0, t\}\omega_{t,1}\{t, 1\}, t]\{0, \tfrac{1}{2}\}\omega_{t,t}\{\tfrac{1}{2}, d\}\beta\{d, 1\}$$
$$= \lambda[\omega, t]\{0, \tfrac{1}{2}\}\omega_{t,t}\{\tfrac{1}{2}, d\}\beta\{d, 1\}$$
$$T(\alpha) = \lambda[\omega, t]\{0, \tfrac{1}{2}\}\beta\{\tfrac{1}{2}, 1\}.$$

(A.2) *Proof of Lemma* (4.8).

We shall first prove that if X is n-connected, $n \geqq 2$, then $d'_*\colon \pi_i(M, S\Omega) \to \pi_i(M \times M, M \vee M)$ is an isomorphism for $i < 3n + 1$ and an epimorphism

for $i = 3n+1$. Consider the following diagram (since all spaces are 1-connected, we may neglect base-points):

(A.3)



Here $u: S\Omega \to S\Omega \vee S\Omega$ shrinks the equator $\Omega \subset S\Omega$ to a point, so that $j'u = i_1 + i_2$, where $j': S\Omega \vee S\Omega \to S\Omega \times S\Omega$ is the inclusion. It is well-known that the lower two sequences split; f and g are the canonical right inverses for j'_* and j''_* respectively. Finally, H is defined by the equation $u_* = \partial' H + f d_*$. The diagram does not commute, but we indicate such commutativity relations as exist.

Note first that $(j \vee j)u = d' \mid S\Omega: S\Omega \to M \vee M$. Hence the two large squares commute: $(j \vee j)_* u_* \partial = \partial'' d'_*$ and $j''_*(j \vee j)_* u_* = d_* j_*$. Squares 2, 3 and 4 clearly are commutative; we shall show that 1 is also.

$(j \vee j)_* u_* \partial = (j \vee j)_* f d_* \partial + (j \vee j)_* \partial' H \partial$; however,

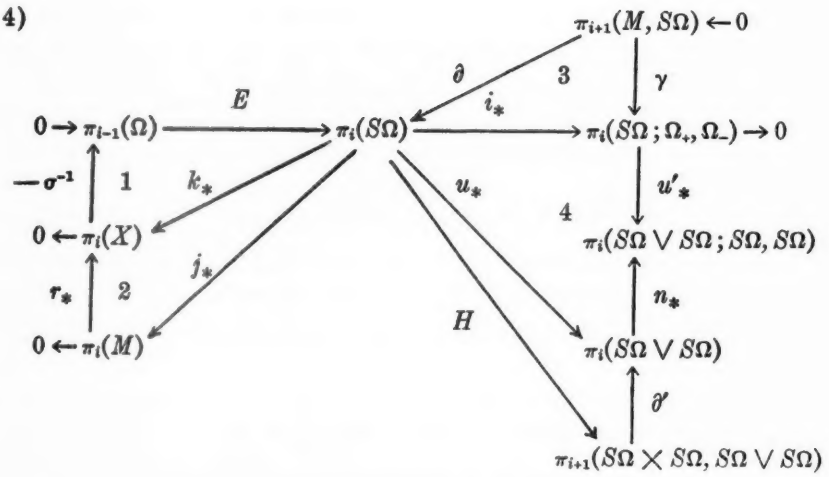
$(j \vee j)_* f d_* \partial = g(j \times j)_* d_* \partial = g d_* j_* \partial = 0$. Hence

$(j \vee j)_* u_* \partial = (j \vee j)_* \partial' H \partial = \partial''(j \times j)_* H \partial$.

But $(j \vee j)_* u_* \partial = \partial'' d'_*$; hence $\partial'' d'_* = \partial''(j \times j)_* H \partial$, from which it follows that $d'_* = (j \times j)_* H \partial$, since ∂'' is a monomorphism. Therefore square 1 is commutative.

We shall next show that $H\partial$ and $(j \times j)_*$ are isomorphisms under suitable conditions; this will imply that d'_* is also an isomorphism. Consider the following diagram:

(A. 4)



The horizontal exact sequence is part of the triad suspension sequence [7]; since by (2.7) E is a monomorphism, i_* is an epimorphism. n_* is the injection, and γ is defined to be $i_*\partial$. The numbered triangles and squares are commutative.

(A. 5) LEMMA. *γ is an isomorphism.*

Proof. 1) kernel γ = kernel $i_* \cap$ image ∂ = image $E \cap$ kernel k_* by exactness. But image $E \cap$ kernel $k_* = 0$, since $-\sigma^{-1}k_*E =$ identity. Hence kernel $\gamma = 0$.

2) Let $a \in \pi_i(S\Omega; \Omega_+, \Omega_-)$. Since i_* is an epimorphism, there exists $a' \in \pi_i(S\Omega)$ such that $i_*(a') = a$. Let $a'' = a' + E\sigma^{-1}k_*(a')$. Then

$$i_*(a'') = i_*(a') + i_*E\sigma^{-1}k_*(a') = a. \quad \text{Also,}$$

$$k_*(a'') = k_*(a') + k_*E\sigma^{-1}k_*(a') = k_*(a') - k_*(a') = 0;$$

hence there exists $b \in \pi_{i+1}(M, S\Omega)$ such that $\partial(b) = a''$, and therefore $\gamma(b) = i_*\partial(b) = i_*(a'') = a$. Therefore γ is an epimorphism.

(A. 6) LEMMA. *$H\partial$: $\pi_{i+1}(M, S\Omega) \rightarrow \pi_{i+1}(S\Omega \times S\Omega, S\Omega \vee S\Omega)$ is an isomorphism for $2 < i+1 \leqq 3n$ and an epimorphism for $i+1 = 3n+1$.*

Proof. Blakers and Masey have shown ([2], Corollary 8.3) that $n_*\partial'$ is an isomorphism for $i+1 > 2$; hence it suffices to consider $n_*\partial' H\partial$. Note that $n_*f = 0$. Hence

(A. 7) $$n_*\partial' H\partial = n_*u_*\partial - n_*fd_*\partial = n_*u_*\partial = u'_*i_*\partial = u'_*\gamma.$$

Let Ω^* be the cone on Ω, $S\Omega^* = S\Omega \cup \Omega^*$ such that $S\Omega \cap \Omega^* = \Omega$, and let $\Omega_+{}^*$, $\Omega_-{}^*$ be the subspaces $\Omega_+ \cup \Omega^*$, $\Omega_- \cup \Omega^*$ of $S\Omega^*$. It follows from Corollary (3.5) of [10] that the injection

$$u''_*: \pi_i(S\Omega; \Omega_+, \Omega_-) \to \pi_i(S\Omega^*; \Omega_+{}^*, \Omega_-{}^*)$$

is an isomorphism for $i < 3n$, and an epimorphism for $i = 3n$.

We denote points of $u(\Omega_-) \subset S\Omega \vee S\Omega$ by $u[\omega, t] = [\omega, 2t]$, $0 \leqq t \leqq \frac{1}{2}$, and those of $u(\Omega_+)$ by $u[\omega, t] = [\omega, 2t-1]'$, $\frac{1}{2} \leqq t \leqq 1$. Points of $\Omega^* \subset S\Omega^*$ will be denoted $[\omega, s]$, $\frac{1}{2} \leqq s \leqq 1$, with $[\omega, 1] = [\omega', 1]$ for all $\omega, \omega' \in \Omega$. Define maps

$$S\Omega^* \xrightarrow{\alpha} S\Omega \vee S\Omega \xrightarrow{\beta} S\Omega^*$$

by $\alpha \mid S\Omega = u$, $\alpha(\Omega^*) = u(\Omega)$,

$$\begin{aligned} \beta[\omega, t] &= [\omega, t] \in \Omega_- && \text{for } 0 \leqq t \leqq \tfrac{1}{2} \\ &= [\omega, t] \in \Omega^* && \text{for } \tfrac{1}{2} \leqq t \leqq 1; \\ \beta[\omega, t]' &= [\omega, 1-t] \in \Omega^* && \text{for } 0 \leqq t \leqq \tfrac{1}{2} \\ &= [\omega, t] \in \Omega_+ && \text{for } \tfrac{1}{2} \leqq t \leqq 1. \end{aligned}$$

Then it is easy to see that $\alpha\beta \cong$ identity, $\beta\alpha \cong$ identity, and that α is a homotopy equivalence of triads $\alpha: (S\Omega^*; \Omega_+{}^*, \Omega_-{}^*) \to (S\Omega \vee S\Omega; S\Omega, S\Omega)$. Hence $\alpha_*: \pi_i(S\Omega^*; \Omega_+{}^*, \Omega_-{}^*) \to \pi_i(S\Omega \vee S\Omega; S\Omega, S\Omega)$ is an isomorphism for all i. Since $\alpha u'' = u'$, it follows that u'_* is an isomorphism for $i < 3n$ and an epimorphism for $i = 3n$. Since γ is an isomorphism, (A.7) implies that $n_*\partial' H\partial$, and hence $H\partial$, is an isomorphism for $i + 1 \leqq 3n$, and an epimorphism for $i + 1 = 3n + 1$. This completes the proof of (A.6).

(A.8) LEMMA. *$(j \times j)_*: \pi_i(S\Omega \times S\Omega, S\Omega \vee S\Omega) \to \pi_i(M \times M, M \vee M)$ is an isomorphism for $i \leqq 3n + 1$.*

Proof. We shall first prove that $(j \times j)_*$ is an isomorphism in homology, and then use a standard method to transfer the result to homotopy. Consider

$$\begin{array}{ccccc} H_i(M \times M, M \vee M) & \xrightarrow{a_*} & H_i(M', (M \vee M)^*) & \xleftarrow{b_*} & H_i(M') \\ \uparrow{\scriptstyle (j \times j)_*} & & \uparrow & & \uparrow{\scriptstyle (j \times j)'_*} \\ H_i(S\Omega \times S\Omega, S\Omega \vee S\Omega) & \xrightarrow{a'_*} & H_i(S\Omega', (S\Omega \vee S\Omega)^*) & \xleftarrow{b'_*} & H_i(S\Omega') \end{array} \tag{A.9}$$

where $S\Omega' = S\Omega \times S\Omega \cup (S\Omega \vee S\Omega)^*$, and similarly for M', and where a_* and b_* are injections. a_* is an isomorphism for all i by excision, b_* for $i > 0$ by exactness. Since there are contractible neighborhoods of the points p_+ and p_- in $S\Omega$ and in M, we may apply the relative Künneth formula: since $j_*: H_i(S\Omega, p_\pm) \to H_i(M, p_\pm)$ is an isomorphism for $i \leqq 2n$ and an epimorphism for $i = 2n+1$, the formula implies that $(j \times j)_*$ is an isomorphism for $i \leqq 3n+1$ and an epimorphism for $i = 3n+2$. By commutativity of (A. 9), the same result therefore holds for $(j \times j)'_*$. Hence by exactness, $H_i(M', S\Omega') = 0$ for $i \leqq 3n+2$. Using the relative Hurewicz theorem, $\pi_i(M', S\Omega') = 0$ for $i \leqq 3n+2$, and by exactness $(j \times j)'_*: \pi_i(S\Omega') \to \pi_i(M')$ is an isomorphism for $i \leqq 3n+1$. Now if we rewrite diagram (A. 9) for homotopy, then the corresponding injection b_* is an isomorphism for all i, while by Corollary (3. 3) of [10], a_* is an isomorphism for $i \leqq 3n+1$. Hence $(j \times j)_*$ is an isomorphism for $i \leqq 3n+1$, which proves (A. 8).

It follows from (A. 6), (A. 8), and the commutativity relations in (A. 3) that $d'_*: \pi_i(M, S\Omega) \to \pi_i(M \times M, M \vee M)$ is an isomorphism for $i < 3n+1$ and an epimorphism for $i = 3n+1$. By reversing the standard argument used in the proof of (A. 8) to transform the homology isomorphism into a homotopy isomorphism, we conclude that (4. 8) holds.

Part II. Eilenberg-MacLane Spaces.

1. Introduction. In order to replace CW-complexes by Postnikov systems, one must know the effect which standard operations on complexes have on the corresponding Postnikov invariants. This is known for the loop space [16], but seems more complicated for suspension. The following proposition determines the invariants of the suspension within the stable range:

Let X be an n-connected CW-complex, $n \geqq 1$, let $q \leqq 2n-1$, and let X_q, $(SX)_{q+1}$ be the q-th and $(q+1)$-st spaces in Postnikov systems for X and SX respectively (cf. § 3 below); then there exists a $(2n+2)$-equivalence $h: S(X_q) \to (SX)_{q+1}$. We identify $\pi = \pi_{q+1}(X)$ with $\pi_{q+2}(SX)$ and $\pi_{q+1}(\Omega SX)$ under E and σE.

(1. 1) PROPOSITION. *If $k^{q+3} \in H^{q+3}((SX)_{q+1}; \pi)$ is a Postnikov invariant for SX, then $-E^*h^*k^{q+3}$ is an invariant for X.*

Proof. The following diagram of isomorphisms is commutative, where h' is induced by h, $i: X_q \to \Omega S(X_q)$ is the standard embedding and all coefficients are π:

$$\begin{array}{ccccc} & & -E^* & & h^* \\ H^{q+2}(X_q) & \longleftarrow & H^{q+3}(S(X_q)) & \longleftarrow & H^{q+3}((SX)_{q+1}) \\ & \nwarrow^{i^*} & \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \sigma^*} \\ & & H^{q+2}(\Omega S(X_q)) & \xleftarrow{h'^*} & H^{q+2}(\Omega((SX)_{q+1})) \end{array}$$

$\sigma^* k^{q+3}$ is an invariant for ΩSX [16]; since $h'i\colon X_q \to \Omega((SX)_{q+1})$ is a homotopy equivalence, $i^* h'^* \sigma^* k^{q+3} = -E^* h^* k^{q+3}$ is also an invariant for ΩSX. Since the embedding $i\colon X \to \Omega SX$ is a $2n$-equivalence, a Postnikov system up to dimension $2n$ (i.e., up to X_{2n-1}, k^{2n+1}) for ΩSX is also a system for X. This proves the proposition.

The simplest category in which to investigate the non-stable case would seem to be that of Eilenberg-MacLane spaces. Our aim is to apply the results of Part I to the computation of some of the invariants of such spaces.

According to [9], the category of spaces having the homotopy type of CW-complexes is closed under the operations of Part I; we shall assume that all spaces which we consider are in this category. Let $K(\pi, n+1)$ be a CW-complex which is an Eilenberg-MacLane space of type $(\pi, n+1)$ (briefly, a $(\pi, n+1)$-space) and let $K = \Omega K(\pi, n+1)$; then K is a (π, n)-space.

2. Homotopy groups of SK. The fundamental map $k\colon SK \to K(\pi, n+1)$ is equivalent to a fibre map in which the fibre has the homotopy type of $K * K$; hence there is an exact sequence, which splits (I, (2.7)):

$$0 \to \pi_i(K * K) \to \pi_i(SK) \xrightarrow{k_*} \pi_i(K(\pi, n+1)) \to 0. \tag{2.1}$$

Consequently we have

(2.2) PROPOSITION.

$$\pi_i(SK) = 0,\ i \leqq n;\ \pi_{n+1}(SK) \approx \pi;\ \pi_i(SK) \approx \pi_i(K * K),\ i \geqq n+2.$$

(2.3) COROLLARY.

$$\pi_i(SK) = 0,\ n+1 < i < 2n+1;\ \pi_{2n+s+1}(SK) \approx H_{n+s}(\pi, n; \pi),\quad 0 \leqq s \leqq n-1.$$

Proof. Since K is $(n-1)$-connected, $K * K$ is $2n$-connected, and the first statement follows. The second statement follows from (2.2) and Theorem 6.1 of [19].

In particular, $\pi_{2n+1}(SK) \approx \pi \otimes \pi$, $\pi_{2n+2}(SK) \approx \pi * \pi$.

3. Moore-Postnikov systems [11]. In this section, let $f: E \to B$ be a fibre map in which B and the fibre F are 1-connected.

(3.1) *Definition.* A sequence of spaces and maps (E_n, g_n, h_n) is a *Moore-Postnikov system* [11] for the fibre map f if:

(1) $g_n: E_n \to E_{n-1}$ is a fibre map with fibre a $(\pi_n(F), n)$-space.

(2) $h_n: E \to E_n$ is an n-equivalence.

(3) $E_0 = B$, $h_0 = f$, and $h_n = g_{n+1}h_{n+1}$.

We denote the composition $g_1 g_2 \cdots g_n: E_n \to B$ by f_n, and set $F_n = f_n^{-1}(b)$, where $b \in B$.

(3.2) *Definition.* A *Postnikov system* [13] for a space E is a Moore-Postnikov system for the trivial fibre map $f: E \to B$, where B is a single point.

(3.3) THEOREM. *Any fibre map has a Moore-Postnikov system.*

The theorem is a consequence of the existence of semi-simplicial Moore-Postnikov systems ([11], Theorem 2.12) and of results of Barratt, Gugenheim and Moore on the geometric realization of certain classes of semi-simplicial fibre spaces (to appear).

Using (3.3), one readily verifies that if F is m-connected, then a Moore-Postnikov system (E_n, g_n, h_n) may be constructed such that $E_i = B$, $h_i = f$, $g_i =$ identity, $i \leqq m$.

(3.4) *Definition.* Let (E_n, g_n, h_n) be a Postnikov system for a space E, $\mathfrak{F}_{n+1}$ the fibre of $g_{n+1}: E_{n+1} \to E_n$, and $b^{n+1} \in H^{n+1}(\mathfrak{F}_{n+1}, \pi_{n+1}(\mathfrak{F}_{n+1}))$ the basic class. Then if τ is the transgression in the fibering g_{n+1},

$$-\tau b^{n+1} \in H^{n+2}(E_n, \pi_{n+1}(\mathfrak{F}_{n+1}))$$

is denoted by $k^{n+2}(E)$ and is called an $(n+2)$nd *Postnikov invariant* of E.

Let Φ be the group of automorphisms of $H^{n+2}(E_n, \pi_{n+1}(\mathfrak{F}_{n+1}))$ determined by homotopy equivalences of E_n and automorphisms of $\pi_{n+1}(\mathfrak{F}_{n+1})$, and let $\{k^{n+2}\}$ be the equivalence class of k^{n+2} under Φ. We identify $\pi_{n+1}(F_{n+1}) \approx \pi_{n+1}(E_{n+1})$ under injection.

(3.5) PROPOSITION. *Let (E_n, g_n, h_n), (E'_n, g'_n, h'_n) be two Postnikov systems for a space E. Then*

(1) *They are equivalent in the sense that there exists a sequence of homotopy equivalences $\lambda_n: E_n \to E'_n$.*

(2) $\lambda_{n+1}{}^{*-1}\lambda_n{}^*k'^{n+2} \in \{k^{n+2}\}$, *where* $\lambda_{n+1}{}^* : \pi_{n+1}(\mathcal{F}_{n+1}) \to \pi_{n+1}(\mathcal{F}'_{n+1})$ *is the coefficient isomorphism.*

(3) *Given* $l \in \{k^{n+2}\}$, *there exists a Postnikov system* (E''_i, g''_i, h''_i) *for* E *in which* $E''_i = E_i$, $g''_i = g_i$, and $h''_i = h_i$, $i \leqq n$, *and such that* $l = -\tau b^{n+1}$.

This proposition is an analogue for fibre spaces of a theorem of Adams[1] on self-obstructions; we omit the proof. It follows that the invariant object is actually the class $\{k^{n+2}\}$, although we shall continue to speak of the elements k^{n+2}. It also follows that we may calculate Postnikov invariants modulo coefficient isomorphisms.

4. A Postnikov system for $SK(\pi, n)$. Consider the fibre map $p : E \to M$, with fibre F, described in Part I, § 3, where $E = E_{S\Omega, M}(M)$, M is the mapping cylinder of $k : S\Omega \to X$ and $X = K(\pi, n+1)$ is a CW-complex. Let (E_i, g_i, h_i) be a Moore-Postnikov system for E in which $E_i = M$, $g_i =$ identity, and $h_i = p$, $i \leqq n+1$. Note that a Postnikov system (E'_i, g'_i, h'_i) for E is obtained by setting

$$E'_i = E_i,\ g'_{i+1} = g_{i+1},\ h'_i = h_i \text{ for } i \geqq n+1;$$

$$E'_i = *,\ g'_{i+1} = *,\ h'_i = *, \quad i < n+1,$$

where $*$ denotes a space consisting of a single point or the map into $*$. Since $E \equiv SK(\pi, n)$,[3] this is also a Postnikov system for the latter space.

(4.1) THEOREM.[4] $k^{i+1}(F) = 0$ *for* $i \leqq 3n$, *so that* $F_{3n} \equiv \prod_{j=1}^{3n} K(\pi_j(F), j)$.

(4.2) LEMMA. [8] *If Y is a 1-connected space, then $k^{i+1}(Y) = 0$ if and only if the Hurewicz homomorphism $\mathcal{H} : \pi_i(Y) \to H_i(Y)$ is a monomorphism onto a direct summand.*

(4.3) LEMMA. $\mathcal{H} : \pi_i(K * K) \to H_i(K * K)$ *is a monomorphism onto a direct summand for* $i \leqq 3n$, *where K is a (π, n)-space.*

The proof of (4.3) will be given later. Since $F \equiv K * K$, (4.1) follows from (4.2) and (4.3).

For the statement and proof of (4.4), i is to be a fixed positive integer, $i \leqq 3n$. Consider the following commutative diagram, in which the notation is that of § 3:

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[3] We write $X \equiv Y$ to indicate that X and Y have the same homotopy type.

[4] This is a generalization of our earlier results, which was noted after the submission of the original version of the paper; it was also remarked by the referee, whom the authors wish to thank for several useful suggestions.

$$\begin{array}{ccccc} & h_i & & = & \\ E & \longrightarrow & E_i & \longleftarrow & E_i \\ \Big\downarrow p & & \Big\downarrow p_i & & \Big\downarrow g_i \\ M & \longrightarrow & M & \longleftarrow & E_{i-1} \\ & = & & p_{i-1} & \end{array}$$

h_i induces by restriction to the respective fibres a map $h'_i\colon F \to F_i$ which is an i-equivalence, while p_{i-1} induces similarly a map $p_{i-1}'\colon \mathfrak{F}_i \to F_i$ such that $p_{i-1}'_*\colon \pi_i(\mathfrak{F}_i) \to (F_i)$ is an isomorphism. Since $\mathfrak{H}$ is natural, $p_{i-1}'_* H_i(\mathfrak{F}_i) = \mathfrak{H}\pi_i(F_i)$.

Choose a left inverse $\mathfrak{H}_F^-$ for the Hurewicz homomorphism $\mathfrak{H}_F\colon \pi_i(F) \to H_i(F)$ and let $a^i(F) \in H^i(F, \pi_i(F))$ be an element such that $\alpha(a^i(F)) = \mathfrak{H}_F^-$, where $\alpha\colon H^i(F, \pi_i(F)) \to \mathrm{Hom}(H_i(F), \pi_i(F))$ is the homomorphism of the universal coefficient theorem. We shall initially compute $k^{i+1}(E)$ in terms of this class.

(4.4) PROPOSITION. *For $i \leqq 3n$, Postnikov invariants for E are given by $k^{i+1}(E) = -p_{i-1}{}^*\tau a^i(F) \in H^{i+1}(E_{i-1}, \pi_i(F))$, where τ is the transgression in the fibre space (E, M, p).*

Proof. Let τ_1, τ_2 be the transgressions in the fiberings $E_i \to M$ and $E_i \to E_{i-1}$ respectively. We identify $\pi_i = \pi_i(F) = \pi_i(F_i) = \pi_i(\mathfrak{F}_i)$ under h'_{i*} and p'_{i-1*}. Consider the following diagram, in which we abbreviate $h' = h'_i$ and $p' = p'_{i-1}$:

$$\begin{array}{ccccc} \mathrm{Hom}(H_i(\mathfrak{F}_i), \pi_i) & \xleftarrow{\alpha} & H^i(\mathfrak{F}_i, \pi_i) & \xrightarrow{\tau_2} & H^{i+1}(E_{i-1}, \pi_i) \\ \Big\uparrow \mathrm{Hom}(p'_*, 1) & \approx & \Big\uparrow p'^* & & \Big\uparrow p_{i-1}{}^* \\ \mathrm{Hom}(H_i(F_i), \pi_i) & \xleftarrow{\alpha} & H^i(F_i, \pi_i) & \xrightarrow{\tau_1} & H^{i+1}(M, \pi_i) \\ \Big\downarrow \mathrm{Hom}(h'_*, 1) & \approx & \Big\downarrow h'^* & \approx & \Big\downarrow = \\ \mathrm{Hom}(H_i(F), \pi_i) & \xleftarrow{\alpha} & H^i(F, \pi_i) & \xrightarrow{\tau} & H^{i+1}(M, \pi_i). \end{array}$$

Under the identifications, $\mathrm{Hom}(p'_*, 1)\mathrm{Hom}(h'_*, 1)^{-1}\mathfrak{H}_F^- = \mathfrak{H}^{-1}$. Hence $p'^* h'^{*-1} a^i(F) = b^i(\mathfrak{F}_i)$, and

$$-p_{i-1}{}^*\tau a^i(F) = -\tau_2 p'^* h'^{*-1} a^i(F) = -\tau_2 b^i(\mathfrak{F}_i) = k^{i+1}(E).$$

This proves (4.4). For the remainded of § 4, let $i < 3n$.

Choose a fixed left inverse $\mathfrak{H}^-$ for

$$\mathfrak{H}\colon \pi_i(X \times X, X \vee X) \to H_i(X \times X, X \vee X);$$

that such exists follows readily from (4.3) and the fact that $X*X \equiv S(X \times X/X \vee X)$. By (4.12) of Part I, $\tau a^i(F) = r^*d^*\theta_3^{*-1}a^i(F)$. Now θ_3 is a composition of isomorphisms, each of which has an analogue in homotopy which commutes with $\mathscr{H}$. Thus we have a commutative diagram

$$\begin{array}{ccc} H_{i+1}(X \times X, X \vee X) & \xrightarrow{\theta_3^{-1}} & H_i(F) \\ \mathscr{H} \uparrow & & \uparrow \mathscr{H}_F \\ \pi_{i+1}(X \times X, X \vee X) & \xrightarrow{\theta_3^{-1}} & \pi_i(F) \end{array}$$

in which θ_3^{-1} is an isomorphism in both cases, and $\mathscr{H}$ and $\mathscr{H}_F$ are monomorphisms onto direct summands. Hence if we define a left inverse $\mathscr{H}_F^-$ by requiring that it commute with $\mathscr{H}^-$ and identifying under θ_3: $\pi_i(F) \approx \pi_{i+1}(X \times X, X \vee X)$, then the dual isomorphism θ_3^{*-1}: $H^i(F; \pi_i(F)) \to H^{i+1}(X \times X, X \vee X; \pi_{i+1}(X \times X, X \vee X))$ carries the coset $\alpha^{-1}(\mathscr{H}_F^-)$ onto the coset $\alpha^{-1}(\mathscr{H}^-)$; i.e., for each $a^{i+1} = a^{i+1}(X \times X, X \vee X)$ there exists $a^i(F) \in \alpha^{-1}(\mathscr{H}_F^-)$ such that $\theta_3^{*-1}a^i(F) = a^{i+1}$. Then from (4.4),

$$k^{i+1}(E) = -p_{i-1}^*\tau a^i(F) = -p_{i-1}^*r^*d^*a^{i+1}.$$

We define a new sequence of fiberings by setting

$$E''_0 = X, \quad E''_i = p_i^{-1}(X) \subset E_i, \quad p''_i = p_i \mid E''_i : E''_i \to X.$$

Since the inclusion i_{i-1}: $E''_{i-1} \subset E_{i-1}$ is a homotopy equivalence, we may replace $k^{i+1}(E)$ by $i_{i-1}^*k^{i+1}(E)$, and the new Postnikov invariant is given by

$$k^{i+1}(E) = -i_{i-1}^*p_{i-1}^*r^*d^*a^{i+1} = -p''^*_{i-1}d^*a^{i+1},$$

since $i_{i-1}^*p_{i-1}^* = p''^*_{i-1}i^*$ and $i^*r^* =$ identity. Hence

(4.5) PROPOSITION. *Postnikov invariants $k^{i+1}(E)$, $i < 3n$, are given by $k^{i+1}(E) = -p''^*_{i-1}d^*a^{i+1}(X \times X, X \vee X)$, where a^{i+1} is any element such that $\alpha(a^{i+1})$ is a left inverse for $\mathscr{H}$.*[5]

The element $a^{2n+2}(X \times X, X \vee X)$, which is the basic class, is easy to calculate. Since the following diagram of isomorphisms is commutative ([2], 5.11), where μ and ν are the homology and homotopy cross-products,

[5] The restriction $i < 3n$ is caused by the dimensional restriction in (4.9) of Part I. If π is finitely-generated, then θ_3 is an epimorphism of the finite group $H_{3n}(F)$ onto an isomorphic group, and therefore an isomorphism. Hence, in this case (4.5) holds for $i \leqq 3n$.

$$\begin{array}{ccc} H_{n+1}(X)\otimes H_{n+1}(X) & \xrightarrow{\mu} & H_{2n+2}(X\times X, X\vee X) \\ \uparrow{\scriptstyle \mathcal{H}\otimes\mathcal{H}} & & \uparrow{\scriptstyle \mathcal{H}} \\ \pi_{n+1}(X)\otimes\pi_{n+1}(X) & \xrightarrow{\nu} & \pi_{2n+2}(X\times X, X\vee X) \end{array}$$

it follows readily that $a^{2n+2}(X\times X, X\vee X) = b\times b$, with coefficient pairing ν, where $b = b^{n+1}(X)$ is the basic class. Identifying under ν, from (4.5) we obtain

(4.6) COROLLARY.

$$k^{2n+2}(SK(\pi,n)) = -d^*(b\times b) = -b\cup b \in H^{2n+2}(X;\pi\otimes\pi),$$

with the natural coefficient pairing.

We shall next obtain some information concerning the reduced join ([7], § 14) $X\,\mathbf{X}\,X = X\times X/X\vee X$ which is needed for (4.10), and in the process shall prove (4.3). Consider the spectral sequence for the homotopy of the join $X*X$, the analogous (trivial) spectral sequence for the homology of $X*X$, and the map of the first into the second induced by the Hurewicz homomorphism ([19], § 5). We are interested in the terms

$$E^1{}_{s-1,0}(\pi) = \pi_{2n+s+2}(X*X^{n+s}, X*X^{n+s-1}) \xrightarrow{\mathcal{H}} H_{2n+s+2}(X*X^{n+s}, X*X^{n+s-1}) = E^1{}_{s-1,0}(H).$$

It follows easily from Lemmas 5.2, 5.3 of [19] and their analogues for homology that the following diagram commutes for all s,

$$\begin{array}{ccc} E^1{}_{s,0}(\pi) & \xrightarrow{d^1} & E^1{}_{s-1,0}(\pi) \\ \mathcal{H}\downarrow & & \downarrow\mathcal{H} \\ E^1{}_{s,0}(H) & \xrightarrow{d^1} & E^1{}_{s-1,0}(H) \end{array}$$

and that the vertical homomorphisms $\mathcal{H}$ are isomorphisms. Hence $\mathcal{H}$ induces isomorphisms $\mathcal{H}_\#: E^2{}_{s,0}(\pi)\to E^2{}_{s,0}(H)$. On the other hand, $E^2{}_{s,0}(H) = E^\infty{}_{s,0}(H)$ as is well-known, and $E^2{}_{s,0}(\pi) = E^\infty{}_{s,0}(\pi)$ for $s\leqq n+1$, by [19], 6.1. Also,

$$E^\infty{}_{s-1,0}(\pi) \approx \pi_{2n+s+2}(X*X)/i'\pi_{2n+s+2}(X*X^{n+s-1})$$

and similarly for homology. By naturality of the Hurewicz homomorphism, we obtain the following commutative diagram for $s\leqq n+1$, where $\mathcal{H}_\#$ is an isomorphism:

$$(4.7)\qquad \begin{array}{ccc} H_{2n+s+2}(X*X) & \xrightarrow{\eta'} & H_{2n+s+2}(X*X)/i'H_{2n+s+2}(X*X^{n+s-1}) \\ \uparrow{\scriptstyle \mathcal{H}} & & \uparrow{\scriptstyle \mathcal{H}_\#} \\ \pi_{2n+s+2}(X*X) & \xrightarrow{\eta''} & \pi_{2n+s+2}(X*X)/i'\pi_{2n+s+2}(X*X^{n+s-1}). \end{array}$$

Here η', η'' are the canonical maps, and η'' is an isomorphism for $s \leqq n+1$.

There is a singular homotopy equivalence $\phi\colon Y*Z \to S(Y \mathbf{X} Z)$ by [19], § 7; applying this to (4.7) and de-suspending (this is justified, since $s \leqq n+1$), we obtain the commutative diagram, where $\mathcal{H}_\#$ and η'' are isomorphisms:

$$(4.8)\qquad \begin{array}{ccc} H_{2n+s+1}(X \mathbf{X} X) & \xrightarrow{\eta'} & H_{2n+s+1}(X \mathbf{X} X)/i'H_{2n+s+1}(X \mathbf{X} X^{n+s-1}) \\ \uparrow{\scriptstyle \mathcal{H}} & & \uparrow{\scriptstyle \mathcal{H}_\#} \\ \pi_{2n+s+1}(X \mathbf{X} X) & \xrightarrow{\eta''} & \pi_{2n+s+1}(X \mathbf{X} X)/i'\pi_{2n+s+1}(X \mathbf{X} X^{n+s-1}). \end{array}$$

One verifies by use of the Künneth formula that η' is the projection onto the direct summand $H_{n+1}(X) \otimes H_{n+s}(X) + H_{n+1}(X) * H_{n+s-1}(X)$.

(4.9) LEMMA. *Let $f\colon A \to B + C$ be a homomorphism of Abelian groups such that the composition $A \to B + C \to B + C/C$ is an isomorphism. Then f is a monomorphism onto a direct summand.*

The proof is elementary.

Since $\mathcal{H}_\#\eta''$ is an isomorphism in (4.8), so is $\eta'\mathcal{H}$; and since η' is the projection onto a direct summand, by (4.9), $\mathcal{H}$ is a monomorphism onto a direct summand for $s \leqq n+1$. Suspending $X \mathbf{X} X$ and using ϕ, one obtains (4.3).

We now proceed to the computation of the elements a^{i+1}, described in (4.5). The identification map $X \times X \to X \mathbf{X} X$ induces isomorphisms $H_{2n+s+1}(X \times X, X \vee X) \approx H_{2n+s+1}(X \mathbf{X} X)$ for all s and $\pi_{2n+s+1}(X \times X, X \vee X) \approx \pi_{2n+s+1}(X \mathbf{X} X)$ for $s \leqq n$. It suffices therefore to compute the corresponding element $a^{i+1}(X \mathbf{X} X) \in H^{i+1}(X \mathbf{X} X); \pi_{i+1}(X \mathbf{X} X))$ such that $\alpha(a^{i+1})$ is a left-inverse for $\mathcal{H}$.

In the Künneth formula,

$$H_{2n+s+1}(X \mathbf{X} X) \approx \sum_{\substack{i+j=2n+s+1 \\ i,j>0}} H_i(X) \otimes H_j(X) + \sum_{\substack{i+j=2n+s \\ i,j>0}} H_i(X) * H_j(X)$$

the embedding of the torsion term $H_i(X) * H_j(X)$ is only well-defined modulo $H_{i+1}(X) \otimes H_j(X) + H_i(X) \otimes H_{j+1}(X)$ ([6], § 12). Since $H_{n+2}(X) = 0$, $H_{2n+s+1}(X \mathbf{X} X)$ splits into two natural direct summands.

$$G = H_{n+1}(X) \otimes H_{n+s}(X) + H_{n+1}(X) * H_{n+s-1}(X),$$

and the sum of the remaining terms. The homomorphism $\mathcal{H}: \pi_{2n+s+1}(X \times X) \to H_{2n+s+1}(X \times X)$ composed with the projection onto G is an isomorphism. Hence, if an element $a \in H^{2n+s+1}(X \times X; \pi_{2n+s+1}(X \times X))$ be found such that $\alpha(a) \in \mathrm{Hom}(H_{2n+s+1}(X \times X); \pi_{2n+s+1}(X \times X))$ is an isomorphism on G and 0 on the other summand, then $\alpha(a)$ is, up to coefficient isomorphism, a left-inverse for $\mathcal{H}$. Since we are calculating modulo coefficient isomorphism, we may replace $\pi_{2n+s+1}(X \times X)$ by G.

Let π be finitely-generated, $\pi = \sum A(u_i)$, where u_i is a generator of $A(u_i) = Z$ or Z_{p^t}, p a prime. We shall see that it suffices to calculate a certain element $a(i,j)$ for each pair (i,j), and to add all $a(i,j)$ in order to obtain a. The notation in the following diagram will be explained in the various cases:

$$\begin{array}{ccc}
H^{2n+s+1}(K' \times K''; G_{ab}) & \xrightarrow{\alpha} & \mathrm{Hom}(H_{2n+s+1}(K' \times K''); G_{ab}) \\
 & \searrow^{\alpha'} & \uparrow \mathrm{Hom}(\lambda_p, 1) \\
\downarrow \lambda_{p*} & & \mathrm{Hom}(H_{2n+s+1}(K' \times K''; Z_p); G_{ab}) \\
 & & \downarrow \mathrm{Hom}(1, \lambda_p) \\
H^{2n+s+1}(K' \times K''; G'_{ab}) & \xrightarrow{\alpha''} & \mathrm{Hom}(H_{2n+s+1}(K' \times K''; Z_p), G'_{ab})
\end{array}$$

We restrict s to $3 \leqq s \leqq n$, and use without further reference the results of [4], § 5.

Case 1.

$$K' = K(Z_{p^a}, n+1), \qquad K''(Z_{p^b}, n+1),$$

$$G_{ab} = H_{n+1}(K') \otimes H_{n+s}(K'') + H_{n+1}(K') * H_{n+s-1}(K'').$$

λ_p denotes reduction modulo p and the induced homomorphism on homology groups. $H_{2n+s+1}(K' \times K'')$ is a Z_p-vector space; it is mapped by λ_p isomorphically onto a direct summand of $H_{2n+s+1}(K' \times K''; Z_p)$, while G_{ab} is mapped isomorphically onto the direct summand $G'_{ab} = H_{n+1}(K'; Z_p) \otimes H_{n+s}(K''; Z_p)$ by $\Lambda_p = \eta\lambda_p$, where η is the projection onto G'_{ab}. $\mathrm{Hom}(1, \Lambda_p)$ is an isomorphism, and so is the homomorphism Λ_{p*} induced by Λ_p. The homomorphisms $\alpha, \alpha', \alpha''$ are given by the universal coefficient theorems. The required element a is $\Lambda_{p*}^{-1}(a')$, where $a' \in H^{2n+s+1}(K' \times K''; G'_{ab})$ is such that $\alpha''(a')$ is the projection onto G'_{ab}. Let $b' \in H^{n+1}(K'; Z_p)$ be the image of the basic class under reduction modulo p, and similarly for b''. Now $H_{n+s}(K''; Z_p) = \sum Z_p(J)$,

where $Z_p(J) \approx Z_p$ and the Steenrod homology operation $St^p{}_J$ maps $Z_p(J)$ isomorphically onto $H_{n+1}(K''; Z_p)$ and vanishes on the other summands ($St^p{}_J$ is the dual of $St^J{}_p = St^I{}_p$ or $St^{I,b}{}_p$ in the notation of Cartan). Hence we may take $a' = b' \times \sum St^J{}_p b''$, where the sum is taken over the Cartan basis of $H^{n+s}(K''; Z_p)$, $St^J{}_p b''$ has coefficients in $Z_p(J)$, and the pairing for the cross-product is the natural map

$$Z_p \otimes Z_p(J) \to H_{n+1}(K'; Z_p) \otimes H_{n+s}(K''; Z_p) = G_{ab}.$$

Case 2. $K' = K(Z_{p^a}, n+1)$, $K'' = K(Z, n+1)$. The discussion for this case is identical with the preceding one, except that no operations of type $St^{I,b}{}_p$ occur.

Case 3.

$$K' = K(Z, n+1),\ K'' = K(Z_{p^b}, n+1),\ G_{ab} = H_{n+1}(K') \otimes H_{n+s}(K'').$$

$H_{2n+s+1}(K' \times K'')$ is again a Z_p-vector space; λ_p maps G_{ab} isomorphically onto the direct summand $G'_{ab} = H_{n+1}(K'; Z_p) \otimes \bar{H} \subset H_{n+1}(K'; Z_p) \otimes H_{n+s}(K''; Z_p)$, where $\bar{H}$ is the subspace of $H_{n+s}(K''; Z_p)$ generated by elements corresponding to those $St^p{}_J$ such that the dual $St^J{}_p = St^I{}_p$ or $St^{I,b}{}_p$ corresponds to a sequence $I = (a_0, \cdots, a_k)$ with $a_k \equiv 0 (2p-2)$. We may therefore take $a' = b' \times \sum St^J{}_p b''$, where $St^J{}_p b''$ has coefficients in $Z_p(J) \subset \bar{H}$, the sum is taken over those elements of the basis for $H^{n+s}(K''; Z_p)$ which do not end in a Bockstein operator, and the pairing of coefficients is the same as in case 1.

Case 4. $K' = K'' = K(Z, n+1)$. $G_{ab} = H_{n+1}(K') \otimes H_{n+s}(K'')$. Here $H_{2n+s+1}(K' \times K'')$ is a torsion group, and its p-primary component $H_{2n+s+1}(K' \times K'')_p$ is a Z_p-vector space; λ_p maps it isomorphically onto a direct summand of $H_{2n+s+1}(K' \times K''; Z_p)$, and $(G_{ab})_p$ isomorphically onto the direct summand G'_{ab}, as in case 3. In order to find an element a such that $\alpha(a)$ is a projection onto G_{ab}, it suffices to find, for each prime p such that $H_{n+s}(Z, n+1)_p \neq 0$, an element a'_p such that $\alpha''(a'_p)$ is a projection onto G'_{ab}. As in case 3, such an element is given by $a'_p = b' \times \sum St^I{}_p b''$, where $St^I{}_p$ has coefficients in $Z_p(I)$, and the sum is taken over those elements of the basis for $H^{n+s}(K''; Z_p)$ which do not end with a Bockstein operator.

We conclude the computation by combining the various cases considered above. If A, A' are Abelian groups,

$$H_r(A + A', n+1) = H_r(A, n+1) + H_r(A', n+1)$$

for $r \leqq 2n+1$; hence, the group $H_{2n+s+1}(X \times X)$ is a bilinear function of the summands $A(u_i)$ of π. In order to calculate the required element a, we

therefore need only consider pairs $A(u_i)$, $A(u_j)$ of summands, and calculate the corresponding $a(i,j)$ as above.

Applying (4.5) and the above computation, we obtain the following. Let $\eta_i\colon \pi \to A(u_i)$ be the projection, b_i the reduction modulo p (where p is the appropriate prime in each case) of $\eta_{i*}b \in H^{n+1}(E_{2n+s-1}, A(u_i))$, where b is the basic class in E_{2n+s-1}.

(4.10) THEOREM.

$$-k^{2n+s+1}(SK(\pi,n)) \in H^{2n+s+1}(E_{2n+s-1};G), \qquad 3 \leqq s \leqq n,$$

is the sum of the following terms, indexed by ordered pairs (u_i, u_j) of generators, not necessarily distinct, in which (a) *the operations $St^J{}_p$ for the case (i,j) has coefficient group $Z_p(i,j,J) = Z_p$, and the coefficient groups with different indexing sets are distinct*; (b) *the cup-product pairings are all isomorphisms*; (c) *G is the direct sum of the coefficient groups for the various terms*:

(1) *If u_i and u_j have orders which are powers of the same prime, say p^a and p^b respectively: a term $b_i \cup \sum St^J{}_p b_j$, where $\{St^J{}_p\}$ is the basis for $H^{n+s}(Z_{p^b}, n+1; Z_p)$.*

(2) *If u_i, u_j have orders p^a, ∞ respectively: a term $b_i \cup \sum St^I{}_p b_j$, where $\{St^I{}_p\}$ is the basis for $H^{n+s}(Z, n+1; Z_p)$.*

(3) *If u_i, u_j have orders ∞, p^b respectively: a term $b_i \cup \sum St^J{}_p b_j$, where $\{St^J{}_p\}$ is the set of those elements of the basis for $H^{n+s}(Z_{p^b}, n+1; Z_p)$ which do not end in a Bockstein operator.*

(4) *If both u_i, u_j have order ∞: for each prime p such that $H_{n+s}(Z, n+1)_p \neq 0$, a term $b_i \cup \sum St^I{}_p b_j$, where $\{St^I{}_p\}$ is the set of those elements of the basis for $H^{n+s}(Z, n+1; Z_p)$ which do not end in a Bockstein operator.*

In the case $s=2$, $H_{2n+s+1}(X \times X)$ is a torsion group, but the p-primary component is no longer a Z_p-vector space. For this case, let $b_i = \eta_{i*}b$, without reducing modulo p. Then, by a simple direct argument, we have

(4.11) THEOREM. $-k^{2n+3}(SK(\pi,n)) = \sum \theta_i b_i \cup \theta_j b_j \in H^{2n+3}(E_{2n+1};G)$, *where* (a) *the summation is over those pairs of generators u_i, u_j whose orders are powers of the same prime, say p^a, p^b respectively*; (b) $\theta_i = 1, \delta_{p^a}, \delta_{p^a}$, $\theta_j = \delta_{p^b}, 1, 1$, *according as $a<b$, $a>b$, $a=b$, where $\delta_{p^k}\colon H^{n+1}(K;Z_{p^k}) \to H^{n+2}(K;Z_{p^k})$ is the Bockstein operator*; (c) *the coefficient pairing $A(u_i) \otimes A(u_j) \to A(u_i) * A(u_j)$ is an isomorphism*; (d) $G = \sum A(u_i) * A(u_j)$.

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AN INTERPOLATION PROBLEM FOR BOUNDED ANALYTIC FUNCTIONS.*

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1. In a classical paper [4], R. Nevanlinna has given a necessary and sufficient condition that the interpolation

$$f(z_\nu) = w_\nu, \qquad \nu = 1, 2, \cdots, \tag{1.1}$$

be possible for a given sequence of points $\{z_\nu\}$, $|z_\nu| < 1$, and an analytic function $f(z)$ in $|z| < 1$, $|f(z)| \leqq 1$. The result is, however, very implicit and gives in a concrete situation very little help in deciding if the interpolation is possible or not.

The object of the present paper is to give a simple and explicit condition on $\{z_\nu\}$ under which the interpolation (1.1) is possible with a bounded function $f(z)$. If we allow $\{w_\nu\}$ to be an arbitrary bounded sequence, the condition is also a necessary one. It should be observed that even the existence of *any* infinite such sequence $\{z_\nu\}$ is non-trivial; this problem was suggested by R. C. Buck and constructions of such examples have also recently been given by Gleason and Newman (unpublished).

The proof of the main theorem depends on a reformulation of problem (1.1) which is presented in section 2. It is essentially included in a result by Garabedian [2]; since the discussion there is quite general and the proof complicated, we have included a complete and simple proof here. Section 3 contains an inequality of the Schwarz type, which is the crucial step in our proof. This is then completed in section 4. The last section is devoted to an application to the ideal structure in the algebra of bounded analytic functions.

2. Let $z_1, z_2, \cdots, z_n$ be a sequence of different complex numbers, $0 < |z_\nu| < 1$, and let $w_1, w_2, \cdots, w_n$ be arbitrary complex numbers. Let H^q, $q \geqq 1$, be the Banach space of functions $f(z)$ analytic in $|z| < 1$ under the norm

$$\| f \|_q = \lim_{r \to 1} \{ \tfrac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^q \, d\theta \}^{1/q}.$$

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We consider the extremal problem

$$\inf_{f \in H^q} \| f \|_q = m_q \tag{2.1}$$

under the conditions

$$f(z_\nu) = w_\nu, \qquad \nu = 1, \cdots, n. \tag{2.2}$$

It is obvious that the lower bound m_q is assumed for a (unique) function $f_q \in H^q$. If $\pi_n(z)$ is a Blaschke product corresponding to the sequence $\{z_\nu\}$,

$$\pi_n(z) = \prod_1^n (z - z_\nu)/(1 - z\bar{z}_\nu), \tag{2.3}$$

then any function in H^q satisfying conditions (2.2) has the form $f = f_q + \eta\pi_n \cdot g$, $g \in H^q$. If η is an arbitrary (small) complex number, a standard variation yields

$$\int_{|z|=1} |f_q|^{q-2} \bar{f}_q \pi_n g \, d\theta = 0, \qquad g \in H^q. \tag{2.4}$$

If we choose for $g(z)$ the functions z^p, $p = 0, 1, \cdots$, the well-known Riesz theorem shows that there is a function $F_q(z) \in H^1$ with boundary values a.e. on $|z| = 1$ such that

$$F_q = |f_q|^{q-2} \bar{f}_q \pi_n z^{-1} m_q^{1-q}. \tag{2.5}$$

If we use the boundary values (2.5), Hölder's inequality yields

$$\| F_q \|_1 \leqq 1. \tag{2.6}$$

We now observe that by Cauchy's integral formula, we have

$$\begin{aligned} m_q = \tfrac{1}{2\pi} m_q^{1-q} \int_{|z|=1} |f_q|^q \, d\theta &= \tfrac{1}{2\pi i} \int_{|z|=1} (F_q \cdot f_q/\pi_n) \, dz \\ &= \sum_1^n F_q(z_\nu) \cdot w_\nu / \pi_n'(z_\nu). \end{aligned} \tag{2.7}$$

We now let $q \to \infty$. By (2.1) and (2.6), we can select a subsequence so that the corresponding sequences $f_q(z)$ and $F_q(z)$ converge to limit functions, $f(z)$ and $F(z)$, which still satisfy conditions (2.2) resp. (2.6). $f(z)$ is also bounded and satisfies (2.2) with the least possible maximum m_∞. If $G(z)$ is an arbitrary function in H^1, $\| G \|_1 \leqq 1$, then by Cauchy's formula,

$$\left| \sum_1^n G(z_\nu) w_\nu / \pi_n'(z_\nu) \right| = \left| \tfrac{1}{2\pi i} \int_{|z|=1} G f \pi_n^{-1} \, dz \right| \leqq m_\infty \| G \|_1 \leqq m_\infty. \tag{2.8}$$

On the other hand, if we let $q \to \infty$ in (2.7), we have equality in (2.8). Hence the following theorem is proved.

THEOREM 1. *Let m_∞ be the least possible maximum of an analytic function in $|z|<1$ satisfying conditions (2.2). Then*

$$m_\infty = \sup_{\|G\|_1=1} |\sum_1^n G(z_\nu) w_\nu/\pi_n'(z_\nu)|. \tag{2.9}$$

3. We shall in this section prove an auxiliary result of some independent interest. It is related to a well-known theorem by Hardy and Littlewood, see [3], p. 291. We use the letter A to denote absolute constants.

THEOREM 2. *Let σ_ν be a sequence of open subintervals of $(0,1)$. Denote by σ_ν' the interval obtained from σ_ν by adjoining to σ_ν equally long intervals on both sides, and let the length of σ_ν be l_ν. Then the conditions*

$$\sum_{\sigma_j \subset \sigma_i'} l_j \leqq C \cdot l_i \tag{3.1}$$

for all i and a fixed constant C, imply

$$\sum_{\nu=1}^\infty (1/l_\nu)(\int_{\sigma_\nu} f(x)\,dx)^2 \leqq A \cdot C \int_0^1 f(x)^2 dx, \tag{3.2}$$

for all square integrable functions f.

We first reduce our theorem to the case when all σ_ν have the form $(k2^{-n}, (k+1)2^{-n})$, where k and n are integers. Every σ_ν is included in the union of at most four equally long intervals of this form of length $< \frac{1}{2} l_\nu$. Let ω_i be the intervals thus obtained and let m_i be the length of ω_i. Condition (3.1) implies

$$\sum_{\omega_j \subset \omega_i} m_j \leqq 4Cm_i = cm_i. \tag{3.3}$$

and it is sufficient to prove that (3.3) implies that (3.2) holds with σ and l replaced by ω and m.

Let $\chi_\nu(x)$ be the characteristic function of ω_ν and introduce the kernel

$$K(x,y) = \sum_1^\infty \chi_\nu(x)\chi_\nu(y)/m_\nu.$$

We have the inequality, $f > 0$,

$$\begin{aligned} \sum_1^\infty (1/m_\nu)(\int_{\omega_\nu} f(x)\,dx)^2 &= \int_0^1 f(x)\,dx \int_0^1 K(x,y) f(y)\,dy \\ &\leqq \{\int_0^1 f(x)^2 dx\}^{\frac{1}{2}} \{\int_0^1 dx (\int_0^1 K(x,y) f(y)\,dy)^2\}^{\frac{1}{2}}. \end{aligned} \tag{3.4}$$

The last factor can be written, $m_{ij} = m(\omega_i \cap \omega_j)$, as the square root of

$$F(x,x) = \sum_{i,j} (m_{ij}/\sqrt{m_i}\sqrt{m_j}) x_i x_j, \tag{3.5}$$

where $x_i = (1/\sqrt{m_i}) \int_{\omega_i} f,x)\, dx$. If we can prove that the quadratic form $F(x,x)$ has a bound $AC \sum_1^\infty x_i^2$, our theorem follows from (3.4).

To be able to discuss the form (3.5), we must carry out the summation in a special way. We say that the index i belongs to the first generation G_1 if ω_i is not included in any larger interval ω_j; if there are several equal intervals, we choose anyone of them as defining a member of the first generation. The second generation G_2 is defined as the collection of first generations corresponding to the intervals associated with G_1. This can evidently be continued until all intervals are exhausted. Similarly, if ω_i is a fixed interval with i belonging to G_k, $G_{i\nu}$, $\nu = 0, 1, 2, \cdots$, denotes the members of $G_{\nu+k}$ corresponding to the subinterval of ω_i.

Let a_ν be the total length of the intervals corresponding to $G_{i\nu}$ for some i. The inequalities (3.3) and our construction imply ($a_0 = m_i$)

$$\sum_{\nu=n}^{\infty} a_\nu \leqq c a_n, \qquad n = 0, 1, \cdots \tag{3.6}$$

$$a_0 \geqq a_1 \geqq a_2 \geqq \cdots \geqq 0. \tag{3.7}$$

These conditions are sufficient for an explicit estimate of a_ν which is contained in the following lemma.

LEMMA 1. *Let $c \geqq 2$ be a positive integer. The conditions* (3.6) *and* (3.7) *imply*

$$a_n \leqq 4(1 - (1/c))^n a_0. \tag{3.8}$$

Let $N > c$ be a fixed index and let $\{a_\nu\}$ be a sequence which maximizes a_N under the conditions (3.6), (3.7) and $a_0 = 1$. Evidently $a_n = 0$, $n > N$, and $a_n = a_N$, $N \geqq n > N - c$. For $n \leqq N - c$, a_n is defined by recursion,

$$\sum_{\nu=n}^{N} a_\nu = c a_n, \qquad n \leqq N - c,$$

which yields ($a_0 = 1$) $a_n = (1 - (1/c))^n$. Hence

$$a_N \leqq a_{N-c} = (1 - (1/c))^N (1 - (1/c))^{-c} < 4(1 - (1/c))^N,$$

which was our assertion.

We can now complete the proof of Theorem 2. For the quadratic form $F(x,x)$ of (3.5), we have the estimate

$$F(x,x)^2 \leqq 4 \cdot \sum_1^\infty x_i^2 \cdot \sum_{i=1}^\infty (1/m_i) \Big(\sum_{\nu=0}^\infty \sum_{j \in G_{i\nu}} \sqrt{m_j}\, x_j \Big)^2.$$

We define $k > 0$ by $k^2 = 1 - (1/4C)$ and find for the last factor, by Schwarz' inequality and the lemma, the estimate

$$\begin{aligned}
&\sum_{i=1}^\infty (1/m_i) \sum_{\nu=0}^\infty k^{-\nu} \Big(\sum_{j \in G_{i\nu}} \sqrt{m_j}\, x_j \Big)^2 \cdot \sum_0^\infty k^\nu \\
&\quad \leqq 1/(1-k) \sum_{i=1}^\infty (1/m_i) \sum_{\nu=0}^\infty k^{-\nu} \sum_{j \in G_{i\nu}} m_j \cdot \sum_{j \in G_{i\nu}} x_j^2 \\
&\quad \leqq 4/(1-k) \sum_{\mu=1}^\infty \sum_{i \in G_\mu} \sum_{\nu=0}^\infty k^\nu \sum_{j \in G_{i\nu}} x_j^2 = 4/(1-k) \sum_{\mu=1}^\infty \sum_{\nu=0}^\infty k^\nu \sum_{j \in G_{\mu+\nu}} x_j^2 \\
&\quad \leqq 4/(1-k) \sum_{j=1}^\infty x_j^2 \cdot \sum_{\nu=0}^\infty k^\nu = (4/(1-k)^2) \sum_1^\infty x_i^2.
\end{aligned}$$

The definition of k now shows that $(1-k)^{-2} \leqq A^2 C^2$ and our theorem is proved.

4. We can now prove our main theorem.

THEOREM 3. *Let $\{z_\nu\}$ be a sequence of complex numbers, $0 < |z_\nu| < 1$. Then the interpolation problem*

$$f(z_\nu) = w_\nu, \quad f(z) \text{ bounded analytic in } |z| < 1, \tag{4.1}$$

is solvable for arbitrary bounded w_ν if and only if

$$\prod_{\nu \neq \mu} |(z_\nu - z_\mu)/(1 - z_\nu \bar{z}_\mu)| \geqq \delta > 0, \qquad \mu = 1, 2, \cdots. \tag{4.2}$$

Remark. If $[z_\nu, z_\mu]$ denotes the hyperbolic distance between z_ν and z_μ the necessary and sufficient condition for an arbitrary simply-connected domain reads:

$$\prod_{\nu \neq \mu} \operatorname{tgh}(2[z_\nu, z_\mu]) \geqq \delta > 0. \tag{4.2.a}$$

For the proof of the necessity we need the following simple lemma.

LEMMA 2. *Let $|a| < 1$ be a fixed complex number. Then*

$$\sup_\phi |\phi(a)| = 1/(1 - |a|^2), \qquad \|\phi\|_1 \leqq 1. \tag{4.3}$$

When discussing the upper bound (4.3), we may assume $\phi(z) \neq 0$ and set $\sqrt{\phi} = \psi(z)$,

$$\psi(z) = \sum_0^\infty c_\nu z^\nu, \qquad \sum_0^\infty |c_\nu|^2 \leqq 1.$$

Hence

$$|\phi(a)| = |\psi(a)|^2 \leqq \sum_0^\infty |c_\nu|^2 \cdot \sum_0^\infty |a|^{2\nu} \leqq 1/(1-|a|^2)$$

with equality for $\phi(z) = (1-|a|^2)/(1-\bar{a}z)^2$.

Let us now assume that every problem (4.1) is solvable. Consider all f satisfying (4.1) and define a norm in the space of bounded sequence $\{w_\nu\} = w$ by

$$N(w) = \inf_f \sup_{|z|<1} |f(z)|$$

for such f's. The space of w's is a Banach space under this norm which implies

$$N(w) \leqq \text{Const.} \sup_\nu |w_\nu|.$$

Theorem 1 then yields

$$\sum_{\nu=1}^n |G(z_\nu)/\pi_n'(z_\nu)| \leqq \text{Const.}, \qquad \|G\|_1 \leqq 1. \tag{4.4}$$

If we choose for G the function ϕ which by Lemma 2 belongs to the point z_μ, we find in particular

$$(1-|z_\mu|^2)|\pi_n'(z_\mu)| = \prod_{\substack{\nu \neq \mu \\ \nu \leqq n}} |(z_\nu - z_\mu)/(1-z_\nu \bar{z}_\mu)| \geqq \delta > 0, \tag{4.5}$$

which was our assertion.

For the converse, it is necessary to prove that (4.4) holds. By (4.5), this is equivalent to the statement

$$\sum_{\nu=1}^\infty (1-|z_\nu|^2)|G(z_\nu)| \leqq \text{Const.}, \qquad \|G\|_1 \leqq 1. \tag{4.6}$$

In proving (4.6), we need only consider functions $G(z)$ in H^1 which do not vanish in $|z|<1$. We may thus replace G in (4.6) by ϕ^2, $\phi \in H^2$. If, finally, $u(z)$ is the real part of ϕ, we need only prove

$$\sum_1^\infty (1-|z_\nu|^2)u(z_\nu)^2 \leqq \text{Const.}, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})^2\, d\theta \leqq 1, \tag{4.7}$$

for positive harmonic function u.

Let $f(\theta) > 0$ be the boundary values of $u(re^{i\theta})$, $f \in L^2(-\pi, \pi)$. We change the definition of f by defining $f(\theta) \equiv 0$, $|\theta| > 2\pi$. Simple estimates of the Poisson kernel show that

$$u(z_\nu) \leqq A \sum_0^\infty (1/4^n\alpha_\nu) \int_{-2^n\alpha_\nu}^{2^n\alpha_\nu} f(\theta+\theta_\nu)\,d\theta, \qquad z_\nu = (1-\alpha_\nu)e^{i\theta_\nu}.$$

By Schwarz's inequality,

$$\alpha_\nu u(z_\nu)^2 \leqq A \sum_0^\infty (1/5^n)\,(1/2^{n+1}\alpha_\nu)\,(\int_{\sigma_\nu^{(n)}} f(\theta)\,d\theta)^2, \tag{4.8}$$

where $\sigma_\nu = \sigma_\nu^{(n)}$ denotes the part of the interval $|\theta-\theta_\nu| < 2^n\alpha_\nu$ which belongs to $(-2\pi, 2\pi)$. To prove Theorem 3 it is, by Theorem 2, sufficient to prove that the intervals $\sigma_\nu^{(n)}$ satisfy the conditions (3.1) with a constant C not exceeding $A4^n$.

Since

$$|(z_\nu - z_\mu)/(1-z_\nu\bar{z}_\mu)|^2 = 1-(1-|z_\nu|^2)(1-|z_\mu|^2)/|1-z_\nu\bar{z}_\mu|^2,$$

our assumption (4.2) implies

$$\alpha_\nu \sum_{\mu=1}^\infty \alpha_\mu/|1-z_\nu\bar{z}_\mu|^2 \leqq \text{Const.}, \qquad \nu = 1, 2, \cdots,$$

which yields

$$\alpha_\nu \sum_{3\alpha_\nu \geqq \alpha_\mu} \alpha_\mu/(\alpha_\nu^2 + (\theta_\nu-\theta_\mu)^2) \leqq \text{Const.}. \qquad \nu = 1, 2, \cdots. \tag{4.9}$$

We now distinguish two cases. If $\sigma_\nu' = (-2\pi, 2\pi)$, in which case $\alpha_\nu > (2\pi/3)2^{-n}$, we find for the left hand side of (3.1) the estimate

$$\sum_{\alpha_\mu > 2^{-n}} 4\pi + \sum_{\alpha_\mu \leqq 2^{-n}} 2^{n+1}\alpha_\mu \leqq A2^n \leqq (A2^n)(2^{n+1}\alpha_\nu).$$

If σ_ν' is a proper sub-interval of $(-2\pi, 2\pi)$, the estimate (4.9) shows that

$$\sum_{\sigma_\mu \subset \sigma_\nu'} 2^{n+1}\alpha_\mu \leqq A4^n \cdot 2^{n+1}\alpha_\nu.$$

Hence (3.1) holds with $C = A4^n$ and the proof is complete.

5. The method used in section 2 together with Theorem 3 give us the possibility of studying the ideal structure of the Banach algebra B of bounded analytic functions in $|z| < 1$ under the usual norm $\|f\| = \sup_{|z|<1} |f(z)|$.

Let $f(z)$ and $g(z)$ be two functions in the Banach algebra C of functions analytic and uniformly continuous in $|z| < 1$. It is an easy consequence of Gelfand's theory on maximal ideals (compare also [1]) that the closed ideal generated by f and g is C if and only if

$$|f(z)| + |g(z)| \geqq \delta > 0, \qquad |z| < 1. \tag{5.1}$$

Under this condition, there exist coefficients $a(z)$, $b(z)$ in C so that

$$af + bg = 1, \qquad |z| < 1. \tag{5.2}$$

We consider the equation (5.2) for all pairs (a, b) of functions in H^q and study the variational problem

$$\inf\{\| a \|_q^q + \| b \|_q^q\} = m_q^q. \tag{5.3}$$

Let (a_q, b_q) be a minimizing pair, and consider the variation $(a_q + \eta hg, b_q - \eta hf)$, $h \in H^q$, for small complex numbers η. As above, we find that

$$(|a_q|^{q-2}\bar{a}_q g - |b_q|^{q-2}\bar{b}_q f) = F_q, \qquad |z| = 1, \tag{5.4}$$

where $F_q \in H^1$ and $F_q(0) = 0$. Let ϕ be an arbitrary function in C. (5.4) together with $a_q f + b_q g = 1$ yields

$$\begin{aligned} 1/2\pi \int_{|z|=1} m_q^{-q}(|a_q|^q + |b_q|^q) g\phi \, |dz| &= (1/2\pi m_q) \int_{|z|=1} |b_q|^{q-2}\bar{b}_q \phi \, |dz|, \\ 1/2\pi \int_{|z|=1} m_q^{-q}(|a_q|^q + |b_q|^q) f\phi \, |dz| &= (1/2\pi m_q) \int_{|z|=1} |a_q|^{q-2}\bar{a}_q \phi \, |dz|. \end{aligned} \tag{5.5}$$

We now let $q \to \infty$. m_q then tends to m, the least possible maximum for coefficients (a, b) satisfying (5.2). After a selection, the formulas (5.5) become

$$\begin{aligned} \int_{|z|=1} g\phi \, dp &= (1/m) \int_{|z|=1} \phi \, d\beta, \\ \int_{|z|=1} f\phi \, dp &= (1/m) \int_{|z|=1} \phi \, d\alpha, \end{aligned} \tag{5.6}$$

where p, α and β have the following properties

$$dp \geqq 0, \int_{|z|=1} dp = 1, \int_{|z|=1} |d\alpha| \leqq 1, \int_{|z|=1} |d\beta| \leqq 1. \tag{5.7}$$

The formulas (5.6) and (5.7) permit us to prove the following theorem.

THEOREM 4. *Let $\pi(z)$ be a Blaschke product with zeros z_ν and let $f(z)$ be an arbitrary function in B such that $|f(z)| + |\pi(z)| \geqq \delta > 0$ in $|z| < 1$. Then*

$$af + b\pi = 1 \tag{5.8}$$

is solvable with coefficients a and b in B if and only if the interpolation

$$a(z_\nu) = 1/f(z_\nu), \qquad \nu = 1, 2, \cdots, a \in B, \tag{5.9}$$

is possible. (5.8) *is thus solvable if* (4.2) *holds.*

The condition (5.9) is obviously necessary. Let us assume that (5.9) holds and consider the pair of functions in C

$$f_n(z) = f(r_n z), \quad r_n < 1, \quad g_n(z) = \prod_1^n (z - z_\nu)/(1 - z\bar{z}_\nu).$$

If r_n is sufficiently close to 1, the interpolation problem $h_n(z_\nu) = f_n(z_\nu)^{-1}$, $\nu = 1, 2, \cdots, n$, has a solution $h_n(z) \in C$, $|h_n(z)| \leqq M$, M independent of n. Let p_n, α_n, β_n be the function of (5.6) and (5.7) associated with f_n and g_n, and let m_n be the corresponding constant m. Our theorem follows if we can prove that

$$m_n = O(1), \qquad n \to \infty. \tag{5.10}$$

Let C_n be the closed linear subspace of C of functions vanishing at $z_1, z_2, \cdots, z_n$. Every function $\phi \in C_n$ has the form $\phi = g_n \cdot \psi$, $\| \phi \| = \| \psi \|$. We define a linear functional $L_n(\phi)$ on C_n:

$$L_n(\phi) = \int_{|z|=1} \phi \, dp_n = (1/m_n) \int_{|z|=1} \psi \, d\beta_n, \qquad \phi \in C_n. \tag{5.11}$$

It follows that $\| L_n \| \leqq 1/m_n$. By the Hahn-Banach theorem, $L_n(\phi)$ can be extended to C so that

$$|L_n(\phi)| \leqq (1/m_n) \| \phi \|, \qquad \phi \in C. \tag{5.12}$$

On the other hand, if $l_n(\phi)$ is the linear functional defined by the first equation of (5.11) for *all* $\phi \in C$, we obviously have for certain constants λ_ν

$$L_n(\phi) - l_n(\phi) = \sum_1^n \lambda_\nu \phi(z_\nu). \tag{5.13}$$

Choosing $\phi \equiv 1$ we find

$$\left| \sum_1^n \lambda_\nu \right| \geqq 1 - (1/m_n);$$

if, however, we choose $\phi = h_n f_n$, $\phi(z_\nu) = 1$, $\nu = 1, 2, \cdots, n$, while the second formula (5.6) yields

$$1 - (1/m_n) \leqq \left| \sum_1^n \lambda_\nu \right| \leqq (1/m_n)(\| h_n \| \cdot \| f_n \| + \| h_n \|) \leqq (M/m_n)(1 + \| f \|).$$

This inequality proves (5.10) and the proof is complete.

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CONTINUITY OF SOLUTIONS OF PARABOLIC AND ELLIPTIC EQUATIONS.*

By J. NASH.

Introduction. Successful treatment of non-linear partial differential equations generally depends on "a priori" estimates controlling the behavior of solutions. These estimates are themselves theorems about linear equations with variable coefficients, and they can give a certain compactness to the class of possible solutions. Some such compactness is necessary for iterative or fixed-point techniques, such as the Schauder-Leray methods. Alternatively, the a priori estimates may establish continuity or smoothness of generalized solutions. The strongest estimates give quantitative information on the continuity of solutions without making quantitative assumptions about the continuity of the coefficients.

The theory of non-linear elliptic equations in two independent variables is fairly well developed. (See [1] for a survey and bibliography.) An essential part is the a priori Hölder continuity estimate for solutions of uniformly elliptic equations, first proved by Morrey in 1938. All methods used to obtain this estimate have been quite special to two dimensions, utilizing, for example, complex analysis and quasi-conformal mappings (see [2]). The restriction to two variables has been due to this use of such special methods; except for the crucial a priori estimate, the theory is extensible (and in large part has been extended) to n dimensions and to parabolic equations. Our results fill this gap, and it should now be possible to build a general theory of non-linear parabolic and elliptic equations, free of dimension restrictions. Strictly speaking, our work needs some generalization to cover equations with lower order terms, systems, etc. This generalization can probably be accomplished fairly quickly.

In this paper, we consider linear parabolic equations of the form

$$\sum_{i,j} \partial[C_{ij}(x_1, x_2, \cdots, x_n, t)\partial T/\partial x_j]/\partial x_i = \partial T/\partial t, \text{ or}$$
$$\nabla \cdot (C(x,t) \cdot \nabla T) = T_t, \tag{1}$$

* Received May 26, 1958.

where the C_{ij} form a symmetric real matrix $C(x,t)$ for each point x and time t. We assume there are universal bounds $c_2 \geqq c_1 > 0$ on the eigenvalues of C so that any eigenvalue θ_ν satisfies $c_1 \leqq \theta_\nu \leqq c_2$. This is the standard "uniform ellipticity" assumption. The continuity estimate for a solution $T(x,t)$ of (1) satisfying $|T| \leqq B$ and defined for $t \geqq t_0$ is

$$(2) \quad |T(x_1,t_1) - T(x_2,t_2)| \leqq BA\{[|x_1 - x_2|/(t_1 - t_0)^{\frac{1}{2}}]^\alpha + [(t_2 - t_1)/(t_1 - t_0)]^{\frac{1}{2}\alpha/(1+\alpha)}\},$$

where $t_2 \geqq t_1 > t_0$. Here A and α are a priori constants which depend only on c_1 and c_2 and the space dimension n. As a corollary of our results on parabolic equations, we obtain a continuity estimate for solutions of elliptic equations. If $T(x)$ satisfies $\nabla \cdot (C(x) \cdot \nabla T) = 0$ in a region R and the same bounds c_1 and c_2 limit the eigenvalues of $C(x)$, then

$$(3) \quad |T(x_1) - T(x_2)| \leqq BA'(|x_1 - x_2|/d(x_1,x_2))^{\alpha/(1+\alpha)},$$

where α is the α of (2) and A' is an a priori constant $A'(n,c_1,c_2)$, and where $|T| \leqq B$ in R and $d(x_1,x_2)$ is the lesser of the distances of the points x_1 and x_2 from the boundary of R.

Our paper is arranged in six parts, each concluding with the attainment of a result significant in itself. Detailed proofs are given and all the results presented in [14] are covered. An appendix states further results, including continuity at the boundary in the Dirichlet problem, a Harnack inequality, and other results, stated without detailed proofs.

General remarks. The open problems in the area of non-linear partial differential equations are very relevant to applied mathematics and science as a whole, perhaps more so than the open problems in any other area of mathematics, and this field seems poised for rapid development. It seems clear, however, that fresh methods must be employed. We hope this paper contributes significantly in this way and also that the new methods used in our previous paper, reference [10], will be of value.

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid. These are a non-linear parabolic system of equations. Also the relationship between this continuum description of a fluid (gas) and the more physically valid statistical mechanical description is not well understood. (See [11], [12], and [13]). An interest in these questions led us to undertake this work. It became clear that nothing could be done about the continuum description of general fluid flow without the ability to handle

non-linear parabolic equations and that this in turn required an a priori estimate of continuity, such as (2).

Probably one should first try to prove a conditional existence and uniqueness theorem for the flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain *gross* types of singularity, such as infinities of temperature or density. (A gross singularity could arise, for example, from a converging spherical shock wave.) A result of this kind would clarify the turbulence problem.

The methods used here were inspired by physical intuition, but the ritual of mathematical exposition tends to hide this natural basis. For parabolic equations, diffusion, Brownian movement, and flow of heat or electrical charges all provide helpful interpretations. Moreover, to us, parabolic equations seem more natural than elliptic ones. It is certainly true in principle that the theory of parabolic equations includes elliptic equations as a specialization, and in applications an elliptic equation typically arises as the description of the steady state of a system which in general is described by a parabolic equation.

In our work, no difference at all appears between dimensions two and three. Only in one dimension would the situation simplify. The key result seems to be the moment bound (13); it opens the door to the other results. We had to work hard to get (13), then the rest followed quickly.

We are indebted to several persons and institutions in connection with this work, including Bers, Beurling, Browder, Carleson, Lax, Levinson, Morrey, Newman, Nirenberg, Stein and Wiener, the Alfred P. Sloan Foundation, the Institute for Advanced Study, M. I. T., N. Y. U., and the Office of Naval Research.

Part I: The Moment Bound.

More than enough is known about linear parabolic equations with variable coefficients to assure the existence of well behaved solutions for equations of the form (1) if we make strong (qualitative) restrictions on the C_{ij} and restrict the class of solutions to be considered. (See [3] through [7].) Therefore we assume: (a) The $C_{ij}(x,t)$ are uniformly C^∞, (b) $C_{ij}(x,t) = \sqrt{c_1 c_2}\,\delta_{ij}$ (Kronecker delta) for $|x| \geqq r_0$, some large constant. We consider only solutions $T(x,t)$ bounded in x for each t for which the solution is defined, i.e., $\max_x |T(x,t)|$ is finite.

Under these restrictions, any bounded measurable function $T(x, t_0)$ of x given at an initial time t_0 determines a unique continuation $T(x, t)$ defined for all $t \geqq t_0$ and C^∞ for $t > t_0$. Moreover, $T(x, t) \to T(x, t_0)$ almost everywhere as $t \to t_0$, and $\max_x |T(x, t)|$ is non-increasing in t. It is also known that fundamental solutions, which we discuss below, exist and have the general properties we state. (See [4], [7].)

After the a priori results are established, a passage to the limit can remove the restrictions on the C_{ij}. This is a standard device in the use of a priori estimates. The Hölder continuity (2) makes the family of solutions equicontinuous and forces a continuous limit (generalized) solution to exist. Furthermore, the maximum principle remains valid and with it the unique continuability of solutions bounded in space. The final result requires only measurability for the C_{ij}, plus the uniform ellipticity condition; and the a priori estimates then hold for the generalized solutions.

The use of fundamental solutions is very helpful with equations of the form (1). Our work is built around step by step control of the properties of fundamental solutions and most of the results concern them directly. A fundamental solution $T(x, t)$ has a "source point" x_0 and "starting time" t_0 and is defined and positive for $t > t_0$. Also, $\int T(x, t)\, dx = 1$ for every $t > t_0$, where dx is the volume element in n-space. As $t \to t_0$, the fundamental solution concentrates around ite source point; $\lim T(x, t)$ is zero unless $x = x_0$, in which case it is $+\infty$. Physically, a fundamental solution represents the concentration of a diffusant spreading from an initial concentration of unit weight at x_0 at time t_0.

All fundamental solutions are conveniently unified in a "characterizing function" $S(x, t, \bar{x}, \bar{t})$. For fixed $\bar{x}$ and $\bar{t}$ and as a function of x and t, S is a fundamental solution of (1) with source point $\bar{x}$ and starting time $\bar{t}$. Dually, for fixed x and t, S is a fundamental solution of the adjoint equation: $\nabla_{\bar{x}} \cdot [C(\bar{x}, t) \cdot \nabla_{\bar{x}} S] = -\partial S / \partial \bar{t}$, where time runs backwards. This duality enables us to use estimates for fundamental solutions in two ways on S.

The dependence of a bounded solution $T(x, t)$ on bounded initial data $T(x, t_0)$ is expressible through S:

$$T(x, t) = \int S(x, t, \bar{x}, t_0)\, T(\bar{x}, t_0)\, d\bar{x}; \tag{4}$$

in particular,

$$S(x_2, t_2, x_0, t_0) = \int S(x_2, t_2, x_1, t_1)\, S(x_1, t_1, x_0, t_0)\, dx_1. \tag{5}$$

These are standard relations. (5) reveals a reproductive property of fundamental solutions.

Now consider a special fundamental solution $T = T(x,t) = S(x,t,0,0)$ with source at the origin and starting time zero. Let

$$E = \int T^2\,dx,$$

then

$$E_t = 2\int TT_t\,dx = 2\int T\nabla\cdot(C\cdot\nabla T)\,dx = -2\int \nabla T\cdot C\cdot\nabla T\,dx,$$

by integration by parts. For any vector V, we have $c_1 |V|^2 \leqq V\cdot C\cdot V \leqq c_2 |V|^2$; therefore

$$-E_t \geqq 2c_1\int |\nabla T|^2\,dx. \tag{6}$$

With (6) and a lower bound for $\int |\nabla T|^2\,dx$ in terms of E, we shall be able to bound E above, obtaining our first a priori estimate. To bound $\int |\nabla T|^2\,dx$ we employ a general inequality valid for any function $u(x)$ in n-space. For our purposes, we assume u is smooth and well behaved at infinity. E. M. Stein gave us the quick proof which follows below.

The Fourier transform of $u(x)$ is

$$v(y) = (2\pi)^{-n/2}\int e^{ix\cdot y}u(x)\,dx.$$

This has the familiar property

$$\int |v|^2 dy = \int |u|^2 dx.$$

The transform of $\partial u/\partial x_k$ is $iy_k v$; hence

$$\int |\partial u/\partial x_k|^2\,dx = \int y_k{}^2 |v|^2\,dy,$$

and

$$\int |\nabla u|^2\,dx = \sum_i \int (\partial u/\partial x_k)^2\,dx = \int |y|^2 |v|^2\,dy.$$

Finally,

$$|v| \leqq (2\pi)^{-n/2}\int |e^{ix\cdot y}|\cdot|u|\,dx = (2\pi)^{-n/2}\int |u|\,dx;$$

therefore, for any $\rho > 0$, we have

$$\int_{|y|\leqq\rho} |v|^2\,dy \leqq (\pi^{n/2}\rho^n/(n/2)!)\{(2\pi)^{-n/2}\int |u|\,dx\}^2, \tag{a}$$

using the formula for the volume of an n-sphere. On the other hand,

$$\int_{|y|>\rho} |v|^2\,dy \leqq \int_{|y|>\rho} |y/\rho|^2 |v|^2\,dy = \rho^{-2}\int |\nabla u|^2\,dx. \tag{b}$$

If we choose the value of ρ minimizing the sum of the two bounds (a) and (b), we obtain a bound on $\int |v|^2\,dy = \int |u|^2\,dx$ in terms of $\int |u|\,dx$ and $\int |\nabla u|^2\,dx$. Solved for $\int |\nabla u|^2\,dx$, this is

$$\int |\nabla u|^2\,dx$$
$$\geqq (4\pi n/(n+2))[(n/2)!/(1+n/2)]^{2/n}[\int |u|\,dx]^{-4/n}[\int |u|^2\,dx]^{1+2/n}.$$

Applying the above inequality with $u = T$, remembering that $\int T dx = 1$, we obtain from (6)

$$-E_t \geqq kE^{1+2/n}.$$

This is the first use of a convention we now establish that k is a generic symbol for a priori constants which depend only on n, c_1, and c_2. Any two instances of k should be presumed to be different constants. Thus, from the above inequality, $(E^{-2/n})_t \geqq k$; hence $E^{-2/n} \geqq kt$ and

$$E \leqq kt^{-n/2}. \tag{7}$$

We used above the qualitative fact $\lim_{t\to 0} E = \infty$.

From this first bound (7) and the identity (5), we obtain

$$T(x,t) = \int S(x,t,\bar{x},t/2)S(\bar{x},t/2,0,0)\,d\bar{x},$$

whence

$$(T(x,t))^2 \leqq \int [S(x,t,\bar{x},t/2)]^2\,d\bar{x} \cdot \int [S(\bar{x},t/2,0,0)]^2\,d\bar{x} \leqq [k(t/2)^{-n/2}]^2.$$

Therefore

$$T \leqq kt^{-n/2}, \tag{8}$$

which is a pointwise bound, stronger than (7).

The key estimate controls the "moment" of a fundamental solution

$$M = \int rT\,dx = \int |x|\,T\,dx.$$

To prove $M \leqq kt^{\frac{1}{2}}$ is our first major goal. This is dimensionally the only possible form for a bound on M. The moment bound is essential to all subsequent parts of this paper.

We also define an "entropy."

$$Q = -\int T \log T\,dx. \tag{9}$$

From (8),

$$Q \geqq \int \min_x [-\log T](T\,dx) \geqq -\log(kt^{-n/2}) \int T\,dx,$$

hence

$$Q \geqq \pm k + \tfrac{1}{2}n \log t \tag{10}$$

because $\int T\,dx = 1$. The sharp result $Q \geqq \frac{1}{2}n \log(4\pi e c_1 t)$ is obtainable from a more sophisticated argument.

Our derivation of a bound on M requires a lower bound on M in terms of Q as a lemma. This inequality, which is $M \geqq ke^{Q/n}$, depends only on the facts $T \geqq 0$, $\int T\,dx = 1$. First observe that for any fixed λ,

$$\min_T (T \log T + \lambda T) = -e^{-\lambda-1}.$$

Let $\lambda = ar + b$, where $r = |x|$ and a and b are any constants, and integrate over space, obtaining

$$\int [T \log T + (ar+b)T]\,dx \geqq -e^{-b-1} \int e^{-ar}\,dx,$$

or

$$-Q + aM + b \geqq -e^{-b-1} a^{-n} D_n,$$

where D_n is the well known constant $2^n \pi^{\frac{1}{2}(n-1)} [\frac{1}{2}(n-1)]!$ related to the gamma-function and the surface of the $(n-1)$-sphere. Now set $a = n/M$ and $e^{-b} = (e/D_n) \cdot a^n$. Then $-Q + n + b \geqq -1$ or $n + 1 \geqq Q + \log(n/D_n) + \log(n/M)$; thus $n \log M + n \geqq Q + n \log n - \log D_n$, finally,

$$M \geqq (n/e\, D_n^{1/n}) e^{Q/n} = ke^{Q/n}. \tag{11}$$

This ingenious proof, due to L. Carleson, gives an optimal constant.

The next inequality is a "dynamic" one, connecting the rates of change with time of M and Q. Differentiating (9),

$$Q_t = -\int (1 + \log T) T_t dx = -\int (1 + \log T) \nabla \cdot (C \cdot \nabla T)\,dx$$
$$= \int \nabla (\log T) \cdot C \cdot \nabla T\,dx$$

after integration by parts. This can be rewritten

$$Q_t = \int \nabla (\log T) \cdot C \cdot \nabla (\log T)\,(T\,dx).$$

Since in general $V \cdot c_2 C \cdot V \geqq V \cdot C^2 \cdot V = |C \cdot V|^2$, where V is a vector, we have

$$c_2 Q_t \geqq \int |C \cdot \nabla (\log T)|^2 (T\,dx) \geqq [\int |C \cdot \nabla \log T|\,(T\,dx)]^2$$
$$\geqq [\int |C \cdot \nabla T|\,dx]^2.$$

Here we used the Schwarz inequality in the form $\int_0^1 f^2\,du \geqq [\int_0^1 f\,du]^2$ with du corresponding to $T\,dx$.

By analogous manipulations,

$$M_t = -\int \nabla r \cdot C \cdot \nabla T\,dx \text{ and } |M_t| \leqq \int |\nabla r|\,|C \cdot \nabla T|\,dx,$$

hence,

$$|M_t| \leqq \int |C \cdot \nabla T|\,dx.$$

Combining inequalities,

$$c_2 Q_t \geqq (M_t)^2. \tag{12}$$

This is a powerful inequality. Q is defined as it is in order to obtain (12).

The three inequalities

$$Q \geqq \pm k + \tfrac{1}{2} n \log t \tag{10}$$

$$M \geqq k e^{Q/n} \tag{11}$$

$$c_2 Q_t \geqq (M_t)^2 \tag{12}$$

and the qualitative fact $\lim M = 0$, as $t \to 0$, suffice by themselves to bound above and below both M and Q, as functions of time. No further reference to the differential equation is needed.

From $M(0) = 0$ and (12),

$$M \leqq \int_0^t (c_2 Q_t)^{\frac{1}{2}} \, dt,$$

whence

$$k e^{Q/n} \leqq M \leqq \int_0^t (c_2 Q_t)^{\frac{1}{2}} \, dt.$$

Now define $nR = Q \mp k - \frac{1}{2} n \log t$ in such a way that $R \geqq 0$ corresponds to (10). Then $Q_t = nR_t + n/2t$, and we obtain

$$k t^{\frac{1}{2}} e^R \leqq M \leqq (nc_2)^{\frac{1}{2}} \int_0^t (1/2t + R_t)^{\frac{1}{2}} \, dt.$$

When a and $a + b$ are positive $(a+b)^{\frac{1}{2}} \leqq a^{\frac{1}{2}} + b/2a^{\frac{1}{2}}$, hence

$$\int_0^t (1/2t + R_t)^{\frac{1}{2}} \, dt \leqq \int_0^t (1/2t)^{\frac{1}{2}} dt + \int_0^t (t/2)^{\frac{1}{2}} R_t \, dt$$

$$\leqq (2t)^{\frac{1}{2}} + R(t/2)^{\frac{1}{2}} - \int_0^t R/(8t)^{\frac{1}{2}} dt \leqq (2t)^{\frac{1}{2}} + R(t/2)^{\frac{1}{2}}.$$

Here we used integration by parts and $R \geqq 0$ in the second and third steps. Applying this result,

$$k t^{\frac{1}{2}} e^R \leqq kM \leqq (2t)^{\frac{1}{2}} + R(t/2)^{\frac{1}{2}},$$

or

$$k e^R \leqq kM/t^{\frac{1}{2}} \leqq 2^{\frac{1}{2}}(1 + \tfrac{1}{2} R).$$

Clearly $k e^R$ increases faster in R than $2^{\frac{1}{2}}(1 + \frac{1}{2}R)$ so that R must be bounded above. Therefore $M/t^{\frac{1}{2}}$ is bounded both above and below:

$$k t^{\frac{1}{2}} \leqq M \leqq k t^{\frac{1}{2}}. \tag{13}$$

If we use best possible constants in (10) and (11), we can obtain

$$b_n(2c_1nt)^{\frac{1}{2}} \leqq M \leqq (2c_2nt)^{\frac{1}{2}}[1+\min(\lambda, (\lambda/2)^{\frac{1}{2}})],$$

where

$$b_n = (n/2t)^{\frac{1}{2}}\{\pi^{\frac{1}{2}}/[\tfrac{1}{2}(n-1)]!\}^{1/n} \geqq 2^{-1/2n}$$

and

$$\lambda = \tfrac{1}{2}\log(c_2/c_1) - \log b_n \leqq (1/2n)\log 2 + \tfrac{1}{2}\log(c_2/c_1).$$

Thus λ is relatively small. Since $b_n \to 1$ as $n \to \infty$, the bounds sharpen with increasing n; indeed, they seem surprisingly sharp. For comparison, $M = (2nct)^{\frac{1}{2}}$ in the simple heat equation where $C_{ij} = c\delta_{ij}$ and $c_1 = c_2 = c$.

Part II: The G Bound.

Here we obtain a result limiting the extent to which a fundamental solution can be very small over a large volume of space near its source point. From this result, we can show there is some overlap, defined as $\int \min(T_1, T_2)dx$, of two fundamental solutions with nearby source points, starting simultaneously.

Let T be $S(x, t, 0, 0)$ and let

$$U(\xi, t) = t^{n/2}T(t^{\frac{1}{2}}\xi, t). \tag{14}$$

This coordinate transformation and renormalization makes $\int U\, d\xi = 1$, where $d\xi$ is the volume element. Furthermore, if μ is the constant such that $M \leqq \mu t^{\frac{1}{2}}$, we have $\int |\xi| U\, d\xi \leqq \mu$. For U, equation (1) transforms to

$$2tU_t = nU + \xi \cdot \nabla U + 2\nabla \cdot (C \cdot \nabla U). \tag{15}$$

Let

$$G = \int \exp(-|\xi|^2)\log(U+\delta)\, d\xi, \tag{16}$$

where δ is a small positive constant. G is sensitive to areas where $|\xi|$ is not large and U is small. These tend to make G strongly negative. We later obtain a lower bound on G of the form

$$G \geqq -k(-\log\delta)^{\frac{1}{2}},$$

valid for sufficiently small δ. This bound limits the possibility for U to be small in a large portion of the region where $|\xi|$ is not large. From $U > 0$ the weak lower bound $G > \pi^{n/2}\log\delta$ follows immediately.

Differentiating (16) with respect to time and using (15), we obtain

$$2tG_t = H_1 + H_2 + H_3,$$

where

$$H_1 = n\int \exp(-|\xi|^2)U/(U+\delta)\,d\xi \geqq 0,$$

$$H_2 = \int \exp(-|\xi|^2)\xi\cdot\nabla\log(U+\delta)\,d\xi$$
$$= -\int \nabla\cdot[\exp(-|\xi|^2)\xi]\log(U+\delta)\,d\xi,$$

by integration by parts, so that

$$H_2 = -\int \exp(-|\xi|^2)(\nabla\cdot\xi)\log(U+\delta)\,d\xi$$
$$+\int \exp(-|\xi|^2)(2|\xi|\nabla|\xi|)\cdot\xi\log(U+\delta)\,d\xi$$
$$= -nG + 2\int \exp(-|\xi|^2)|\xi|^2[\log\delta + \log(1+U/\delta)]\,d\xi,$$

hence,

$$H_2 \geqq -nG + 2\log\delta\int|\xi|^2\exp(-|\xi|^2)\,d\xi \geqq -nG + n\pi^{n/2}\log\delta;$$

finally

$$H_3 = 2\int \exp(-|\xi|^2)\nabla\cdot(C\cdot\nabla U)/(U+\delta)\,d\xi$$
$$= -2\int \nabla[\exp(-|\xi|^2)/(U+\delta)]\cdot C\cdot\nabla U\,d\xi$$
$$= 4\int(\exp(-|\xi|^2)[|\xi|\nabla|\xi|\cdot C\cdot\nabla U]/(U+\delta)\,d\xi$$
$$+2\int \exp(-|\xi|^2)[\nabla U\cdot C\cdot\nabla U]/(U+\delta)^2\,d\xi$$
$$= H_3' + H_3'',$$

where

$$H_3' = 4\int \exp(-|\xi|^2)\xi\cdot C\cdot\nabla\log(U+\delta)\,d\xi,$$

$$H_3'' = 2\int \exp(-|\xi|^2)\nabla\log(U+\delta)\cdot C\cdot\nabla\log(U+\delta)\,d\xi.$$

From the Schwarz inequality,

$$(H_3')^2 \leqq \{4\int \exp(-|\xi|^2)\xi\cdot C\cdot\xi\,d\xi\}$$
$$\times\{4\int \exp(-|\xi|^2)\,\nabla\log(U+\delta)\cdot C\cdot\nabla\log(U+\delta)\,d\xi\}$$
$$\leqq \{4c_2\int|\xi|^2\exp(-|\xi|^2)\,d\xi\}2H_3''$$
$$\leqq (4c_2\cdot\tfrac{1}{2}n\pi^{\frac{1}{2}n})\cdot 2H_3'' = 4nc_2\pi^{\frac{1}{2}n}H_3''.$$

Hence

$$|H_3'| \leqq k(H_3'')^{\frac{1}{2}}.$$

Furthermore,

$$H_3'' \geqq 2c_1\int \exp(-|\xi|^2)|\nabla\log(U+\delta)|^2\,d\xi \tag{17}$$

Combining the lower bounds available for H_1, H_2, and H_3', we obtain

$$(18)\qquad \begin{aligned} 2tG_t &\geqq H_1 + H_2 + H_3'' - |H_3'| \\ &\geqq 0 + (-nG + n\pi^{\frac{1}{2}n}\log\delta) + H_3'' - k(H_3'')^{\frac{1}{2}} \\ &\geqq k\log\delta - nG - k(H_3'')^{\frac{1}{2}} + H_3''. \end{aligned}$$

When we bound H_3'' below in terms of G, (18) will yield a lower bound on G.

A function $f(\xi) = f(\xi_1, \xi_2, \cdots, \xi_n)$ may be expanded in products of Hermite polynomials, of the form $\prod H_{\nu(i)}(\xi_i)$, where the polynomials are defined and orthonormalized so that $\int_{-\infty}^{+\infty} \exp(-s^2) H_\nu(s) H_\lambda(s)\, ds = \delta_{\nu\lambda}$. The identity $dH_\nu(s)/ds = (2\nu)^{\frac{1}{2}} H_{\nu-1}(s)$ obtains, and the coefficients of these products in the similar expansion of, say, $\partial f/\partial\xi_i$ depend very simply on the coefficients in the expansion of f. If $\int \exp(-|\xi|^2) f\, d\xi = 0$, the coefficient of $\prod H_0(\xi_i)$ is zero and we obtain

$$\int \exp(-|\xi|^2) |\nabla f|^2\, d\xi = \sum \int \exp(-|\xi|^2)(\partial f/\partial\xi_i)^2 d\xi \geqq 2\int \exp(-|\xi|^2) f^2\, d\xi.$$

Applying the above, with $f = \log(U+\delta) - \pi^{-n/2}G$, to (17), we obtain

$$(19)\qquad H_3'' \geqq 4c_1 \int \exp(-|\xi|^2)[\log(U+\delta) - \pi^{-n/2}G]^2\, d\xi.$$

The quantity $U^{-1}[\log(U+\delta) - \pi^{-n/2}G]^2$, related to the integrand in (19), is large for very small U, then decreases to zero, rises from zero to a local maximum at $U = U_c$, say, and finally decreases monotonically as $U \to \infty$ for $U \geqq U_c$. (We know $\log\delta - \pi^{-n/2}G < 0$.) The equation for the maximum point U_c is $\log(U_c+\delta) - \pi^{-n/2}G = 2U_c/(U_c+\delta)$, from which $U_c < U_0 = \exp(2 + \pi^{-n/2}G)$. Therefore the quantity under discussion is decreasing for $U \geqq U_0$ The bound (8), $T \leqq kt^{-n/2}$, corresponds to $U \leqq k$. Hence the quantity has a lower bound of the form $k[\log(k+\delta) - kG]^2$ for $U \geqq U_0$. Applying this to (19), we may say

$$H_3'' \geqq 4c_1 \int \exp(-|\xi|^2) k[\log(k+\delta) - kG]^2 U^*\, d\xi,$$

where $U^* = U$ for $U > U_0$ and $U^* = 0$ for $U \leqq U_0$. Thus we are ignoring the contribution to (19) of the region where $U \leqq U_0$ and taking the worst case, U as large as possible, in the remaining region. For sufficiently negative G, the expression $\log(k+\delta) - kG$ will remain positive when δ is omitted, so that $[\log k - kG]^2 < [\log(k+\delta) - kG]^2$, and we can simplify the above inequality on H_3'' to the form

$$(20)\qquad H_3'' \geqq (k - kG)^2 \int \exp(-|\xi|^2) U^*\, d\xi.$$

Let $\lambda = \int U^*\, d\xi$ and observe that $\int |\xi|\, U^*\, d\xi \leqq \int |\xi|\, U\, d\xi \leqq \mu$. Therefore

$$\mu \geqq \int_{|\xi| \geqq 2\mu/\lambda} |\xi| U^* \, d\xi \geqq (2\mu/\lambda) \int_{|\xi| \geqq 2\mu/\lambda} U^* \, d\xi,$$

hence,

$$\int_{|\xi| \geqq 2\mu/\lambda} U^* \, d\xi \leqq \tfrac{1}{2}\lambda, \text{ and so } \int_{|\xi| \leqq 2\mu/\lambda} U^* \, d\xi \geqq \lambda - \tfrac{1}{2}\lambda = \tfrac{1}{2}\lambda.$$

This result can be applied to (20) and yields

$$H_3'' \geqq (k - kG)^2 \cdot \exp(-(2\mu/\lambda)^2)(\tfrac{1}{2}\lambda). \tag{21}$$

This is not effective unless we can bound λ below, or bound $\int \hat{U} \, d\xi = 1 - \lambda$ above, where $\hat{U} = U - U^*$ so that $\hat{U} = 0$ unless $U \leqq U_0$, in which case $\hat{U} = U$. Of course, we know $\int |\xi| \hat{U} \, d\xi \leqq \mu$ because $\hat{U} \leqq U$. Under the moment constraint and the constraint $\hat{U} \leqq U_0$, the maximum of $\int \hat{U} \, d\xi$ is clearly realized by having $\hat{U} = U_0$ for $|\xi| \leqq \rho$ and $\hat{U} = 0$ for $|\xi| > \rho$, where ρ is such that

$$\int |\xi| \hat{U} \, d\xi = \int |\xi| U_0 \, d\xi = [n\pi^{n/2}/(n+1)(n/2)!]\rho^{n+1} U_0 = \mu.$$

This makes

$$1 - \lambda = \int \hat{U} \, d\xi = [\pi^{n/2}/(n/2)!]\rho^n U_0,$$

$$1 - \lambda \leqq U_0 (k\mu/U_0)^{n/(n+1)} \text{ or } 1 - \lambda \leqq k U_0^{1/(n+1)}.$$

If U_0, which is $\exp(2 + \pi^{-n/2} G)$, is small enough, then $1 - \lambda$ is small and λ is bounded below. Thus $\lambda \geqq \frac{1}{2}$, say, for all sufficiently large $-G$. Now from (21), we have

$$H_3'' \geqq (k - kG)^2$$

for sufficiently large $-G$.

Returning to inequality (18) controlling G_t and applying the above result, we can state that *for sufficiently negative* G,

$$\begin{aligned} 2dG/d(\log t) = 2tG_t &\geqq -nG + k \log \delta + (k - kG)^2 - k(k - kG) \\ &\geqq k|G|^2 + k \log \delta. \end{aligned} \tag{22}$$

Let $G_1(c_1, c_2, n)$ be the number such that when $G \leqq G_1$, we know G is small enough to make (22) valid. Let $G_2(c_1, c_2, n, \delta) = -k(-\log \delta)^{\frac{1}{2}}$ be the largest number such that $k|G|^2 + k \log \delta > 0$ for all $G < G_2$. Then $\min(G_1, G_2) = G_3$ is the smallest possible value of G. If we had $G(t_1) = G_3 - \epsilon$, we would have $dG/d(\log t) \geqq \epsilon^*$ for all $t \leqq t_1$, and consequently $G(t) \leqq G(t_1) - \epsilon^* \log(t_1/t)$, which implies $G \to -\infty$ as $t \to 0$. But since $G \geqq \pi^{n/2} \log \delta$, the hypothesis $G(t_1) = G_3 - \epsilon$ is impossible. Our conclusion is $G \geqq G_3$, or

$$G \geqq -k(-\log \delta)^{\frac{1}{2}} \tag{23}$$

for all *sufficiently small values of* δ, because $G_2 \leqq G_1$ and

$$G_3 = G_2 = -k(-\log\delta)^{\frac{1}{2}}$$

when δ is small enough.

Part III: The Overlap Estimate

Let T_1 and T_2 be two fundamental solutions $S(x,t,x_1,0)$ and $S(x,t,x_2,0)$ with nearby sources. Change coordinates, defining $U_1 = t^{n/2}T_1(t^{\frac{1}{2}}\xi,t)$ and $U_2 = t^{n/2}T_2(t^{\frac{1}{2}}\xi,t)$. Let $\xi_1 = x_1/t^{\frac{1}{2}}$ and $\xi_2 = x_2/t^{\frac{1}{2}}$. Here the source of the (renormalized) fundamental solution U_i is ξ_i rather than the origin, which was the source of U in Part II. Taking this into account, we apply (23), obtaining

$$\int \exp(-|\xi-\xi_i|^2)\log(U_i+\delta)\,d\xi = G_i \geqq -k(-\log\delta)^{\frac{1}{2}},$$

where $i = 1$ or 2 and δ must be sufficiently small. We may add the inequalities above and obtain

$$\int \max_i[\exp(-|\xi-\xi_i|^2)]\max_i[\log(U_i+\delta)]\,d\xi + \int \min_i[\exp(-|\xi-\xi_i|^2)]\min_i[\log(U_i+\delta)]\,d\xi \geqq -2k(-\log\delta)^{\frac{1}{2}}$$

in which we form two integrals with sum at least as large as the sum of the original integrals. We abbreviate the above to

$$\int f^* \log(U_{\max}+\delta)\,d\xi + \int \hat{f}\log(U_{\min}+\delta)\,d\xi \geqq -k(-\log\delta)^{\frac{1}{2}}.$$

For the first integral, we observe (assuming $\delta \leqq 1$)

$$\int f^* \log(U_{\max}+\delta)\,d\xi \leqq \int f^*(U_1+U_2)\,d\xi \leqq \int (U_1+U_2)\,d\xi = 2.$$

For the second integral,

$$\begin{aligned}\int \hat{f}\log(U_{\min}+\delta)\,d\xi &\leqq \log\delta \int \hat{f}\,d\xi + \max[\hat{f}]\int \log(1+U_{\min}/\delta)\,d\xi \\ &\leqq w\log\delta + \delta^{-1}\int U_{\min}\,d\xi,\end{aligned}$$

where

$$w = \int \min[\exp(-|\xi-\xi_1|^2), \exp(-|\xi-\xi_2|^2)]\,d\xi.$$

Therefore we obtain

$$2 + w\log\delta + \delta^{-1}\int \min(U_1,U_2)\,d\xi \geqq -k(-\log\delta)^{\frac{1}{2}},$$

or

$$\int \min(T_1,T_2)\,dx = \int \min(U_1,U_2)\,d\xi \geqq \delta[-2 - w\log\delta - k(-\log\delta)^{\frac{1}{2}}]. \tag{24}$$

This is valid for sufficiently small δ, say for $\delta \leqq \delta_1$. Also, there is a value $\delta_2(w)$ such that for $\delta < \delta_2(w)$, the bracketed expression is positive. If we set $\delta = \frac{1}{2}\min(\delta_1, \delta_2)$, the right member of (24) is definitely positive, and we may conclude

$$\int \min(T_1, T_2)\,dx \geqq \phi(|\xi_1 - \xi_2|) \geqq \phi(|x_1 - x_2|/t^{\frac{1}{2}}) \tag{25}$$

because w is a function of $|\xi_1 - \xi_2|$. The function ϕ is decreasing but always positive. It is an a priori function, determined only by c_1, c_2, and n. This inequality (25) is our first estimate on the overlap of fundamental solutions. Its weakness is that we know little about the function ϕ.

Part IV: Continuity in Space.

We can obtain a stronger inequality by iterative use of (25). Observe that

$$\begin{aligned} \tfrac{1}{2}\int |T_1 - T_2|\,dx &= \tfrac{1}{2}\int [T_1 + T_2 - 2\min(T_1, T_2)]\,dx \\ &\leqq 1 - \phi(|x_1 - x_2|/t^{\frac{1}{2}}) = \psi(|x_1 - x_2|/t^{\frac{1}{2}}) \end{aligned} \tag{26}$$

in which we define the function ψ, which is increasing but always less than one.

Let $T_a = \max(T_1 - T_2, 0)$ and $T_b = \max(T_2 - T_1, 0)$ so that $T_a + T_b = |T_1 - T_2|$ and $\int (T_a - T_b)\,dx = \int (T_1 - T_2)\,dx = 0$. Then

$$\int T_a dx = \int T_b dx = A(t) = \tfrac{1}{2}\int |T_1 - T_2|\,dx \leqq \psi(|x_1 - x_2|/t^{\frac{1}{2}}),$$

defining $A(t)$. Let

$$\chi(x, \bar{x}, t) = T_a(x) T_b(\bar{x})/A(t).$$

Let $T_a^*(x', t', t)$ be the bounded solution in x' and t' of (1) defined for $t' \geqq t$ and having the initial value $T_a^*(x, t, t) = T_a(x, t)$. Define T_b^* similarly. Then from (4),

$$\begin{aligned} T_a^*(x', t', t) &= \int S(x', t', x, t) T_a(x, t)\,dx \\ &= \int\int S(x', t', x, t)\chi(x, \bar{x}, t)\,dx\,d\bar{x}, \end{aligned}$$

and

$$\begin{aligned} T_1(x', t') - T_2(x', t') &= T_a^* - T_b^* \\ &= \int\int [(x', t', x, t) - S(x', t', \bar{x}, t)]\chi(x, \bar{x}, t)\,dx\,d\bar{x} \end{aligned}$$

by the superposition principle ($T_1 - T_2$ and $T_a^* - T_b^*$ are both solutions of (1) for $t' \geqq t$, and by definition, $T_a^* - T_b^* = T_1 - T_2$ at $t' = t$.). Integrating this over dx', we obtain

$$\tfrac{1}{2}\int |\, T_1(x',t') - T_2(x',t') \,|\, dx'$$
$$\leqq \int\int\int |\, S(x',t',x,t) - S(x',t',\bar{x},t) \,|\, dx'\, \chi(x,\bar{x},t)\, dx\, d\bar{x},$$

whence

$$A(t') \leqq \int\int \psi(|\, x-\bar{x}\,|/(t'-t)^{\frac{1}{2}})\chi(x,\bar{x},t)\, dx\, d\bar{x} \tag{27}$$

by application of (26). Incidentally, the right member above is

$$\leqq \int\int \chi(x,\bar{x},t)\, dx\, d\bar{x} = A(t),$$

thus $A(t') \leqq A(t)$ when $t' \geqq t$. This inequality (27) is the key to the iterative argument which strengthens (25) and (26).

To begin the iterative argument, we choose any specific number d and let $\epsilon = \phi(d) = 1 - \psi(d)$. (If we were trying to get an explicit formula for the exponent α in (2), we would choose d with regard to an explicit formula for $\phi(d)$ so as to optimize the result.) Let $\sigma = 1 - \epsilon/4$. For each integer ν, let t_ν be the time (or the least time) at which $A(t) = A(t_\nu) = \sigma^\nu$, if t_ν exists. This is in reference to a specific pair, T_1 and T_2, of fundamental solutions. We know, for example, that $t_1 < \tau$, where $\tau = |\, x_1 - x_2\,|^2/d^2$, because $A(\tau) \leqq \psi(|\, x_1 - x_2\,|/\tau^{\frac{1}{2}}) = \psi(d) = 1-\epsilon$ and $\sigma = 1 - \epsilon/4 > 1 - \epsilon$, so that $A(\tau) < A(t_1) = \sigma$.

Let $M_a(t) = \int |\, x - x_0\,|\, T_a dx$, where x_0 is $\frac{1}{2}(x_1 + x_2)$, the midpoint of the line segment joining the source points x_1 and x_2 of the fundamental solutions T_1 and T_2. Define M_b similarly and let $M_\nu = \max[M_a(t_\nu), M_b(t_\nu)]$. We decompose T_a into nearer and farther parts T_a' and $T_a - T_a'$ at each time t_ν as follows: for $|\, x - x_0\,| \leqq 2\sigma^{-\nu}M_\nu$, define $T_a' = T_a$; otherwise $T_a' = 0$. Then $2\sigma^{-\nu}M_\nu \int (T_a - T_a')\, dx \leqq \int |\, x - x_0\,|\,(T_a - T_a')\, dx \leqq \int |\, x - x_0\,|\, T_a dx \leqq M_\nu$, and consequently, $\int (T_a - T_a')\, dx \leqq \frac{1}{2}\sigma^\nu$ and $\int T_a'\, dx \geqq \frac{1}{2}\sigma^\nu$. Define T_b' similarly and define $\chi_\nu'(x,\bar{x}) = \sigma^{-\nu} T_a'(x) T_b'(\bar{x})$. Now, applying (27) with $t = t_\nu$, we can say

$$A(t') \leqq \int\int \psi(|\, x-\bar{x}\,|)/(t'-t_\nu)^{\frac{1}{2}}[\{\chi(x,\bar{x},t_\nu) - \chi_\nu'(x,\bar{x})\} + \chi_\nu'(x,\bar{x})]\, dx d\bar{x}$$
$$\leqq \int\int \{\chi - \chi_\nu'\}\, dx d\bar{x} + \psi(4\sigma^{-\nu}M_\nu/(t'-t_\nu)^{\frac{1}{2}}) \int\int \chi_\nu'\, dx d\bar{x},$$

because when $\chi_\nu' > 0$, we know both $T_a' > 0$ and $T_b' > 0$ so that both $|\, x - x_0\,|$ and $|\, \bar{x} - x_0\,|$ are $\leqq 2\sigma^{-\nu}M_\nu$, and consequently, $|\, x - \bar{x}\,| \leqq 4\sigma^{-\nu}M_\nu$, and we also know that $\chi \geqq \chi_\nu'$ and $\psi < 1$. Proceeding further,

$$A(t') \leqq \int\int \chi\, dx d\bar{x} - [1 - \psi(4\sigma^{-\nu}M_\nu/(t'-t_\nu)^{\frac{1}{2}})] \int\int \chi_\nu'\, dx d\bar{x}$$
$$\leqq \sigma^\nu - [1-\psi]\sigma^{-\nu} \int T_a' dx \int T_b' dx \leqq \sigma^\nu - [1-\psi]\sigma^{-\nu}(\sigma^\nu/2)^2$$
$$\leqq \sigma^\nu[3/4 + 1/4\psi(4\sigma^{-\nu}M_\nu/(t'-t_\nu)^{\frac{1}{2}})].$$

We now set $t' = t_\nu + 16\sigma^{-2\nu}(M_\nu)^2 d^{-2}$, and the argument of ψ above becomes d. Then since $\psi(d) = 1 - \epsilon$, we obtain

$$A(t') \leqq \sigma^\nu[3/4 + 1/4(1-\epsilon)] = \sigma^\nu(1 - \epsilon/4) = \sigma^{\nu+1}.$$

Hence

$$t_{\nu+1} \leqq t' = t_\nu + 16\sigma^{-2\nu}(M_\nu)^2 d^{-2}. \tag{28}$$

This will bound the sequence $\{t_\nu\}$ of times after we obtain a bound on the sequence $\{M_\nu\}$ of moments.

Observe that

$$\begin{aligned} T_a(x', t') &= \max(T_1(x', t') - T_2(x', t'), 0) \\ &= \max(T_a^*(x', t', t) - T_b^*(x', t', t), 0) \leqq T_a^*(x', t', t) \\ &= \int S(x', t', x, t) T_a(x, t)\, dx. \end{aligned}$$

Therefore

$$\begin{aligned} M_a(t') &= \int |x' - x_0|\, T_a(x', t')\, dx' \\ &\leqq \int\int [|x' - x| + |x - x_0|] S(x', t', x, t) T_a(x, t)\, dx dx'; \end{aligned}$$

hence

$$\begin{aligned} M_a(t') \leqq{}& \int |x - x_0|\, T_a(x, t) \int S(x', t', x, t)\, dx' dx \\ &+ \int T_a(x, t) \int |x' - x|\, S(x', t', x, t)\, dx' dx, \end{aligned}$$

or

$$\begin{aligned} M_a(t') &\leqq \int |x - x_0|\, T_a(x, t)\, dx + \mu(t' - t)^{\frac{1}{2}} \int T_a(x, t)\, dx \\ &\leqq M_a(t) + A(t)\mu(t' - t)^{\frac{1}{2}}. \end{aligned}$$

Now let t and t' be t_ν and $t_{\nu+1}$, use a similar estimate for M_b and the definition $M_\nu = \max(M_a(t_\nu), M_b(t_\nu))$, and obtain, by (28),

$$\begin{aligned} M_{\nu+1} &\leqq M_\nu + \sigma^\nu \mu (t_{\nu+1} - t_\nu)^{\frac{1}{2}} \\ &\leqq M_\nu + \sigma^\nu \mu (16\sigma^{-2\nu}(M_\nu)^2 d^{-2})^{\frac{1}{2}} \leqq M_\nu(1 + 4\mu/d). \end{aligned}$$

Now $t_0 = 0$ and $M_0 = M_a(t_0) = M_b(t_0) = \frac{1}{2}|x_1 - x_2|$, because T_1 and T_2 concentrate at x_1 and x_2 as $t \to 0$, and $|x_1 - x_0| = |x_2 - x_0| = \frac{1}{2}|x_1 - x_2|$ since $x_0 = \frac{1}{2}(x_1 + x_2)$. Therefore we have

$$M_\nu \leqq \tfrac{1}{2}|x_1 - x_2|(1 + 4\mu/d)^\nu.$$

With this and (28), the sequence $\{t_\nu\}$ can be bounded:

$$t_{\nu+1} \leqq t_\nu + 16\sigma^{-2\nu}[\tfrac{1}{2}|x_1 - x_2|(1 + 4\mu/d)^\nu]^2 d^{-2}.$$

Hence

$$t_{\nu+1} \leqq 4d^{-2} \mid x_1 - x_2 \mid^2 \sum_{\lambda=0}^{\nu} [(1 + 4\mu/d)/\sigma)]^{2\lambda}.$$

Summing this geometrical series,

$$t_\nu / \mid x_1 - x_2 \mid^2 \leqq 4d^{-2}\{(\sigma^{-2}(1 + 4\mu/d))^{2\nu+2}/[\sigma^{-2}(1 + 4\mu/d) - 1]\} \equiv \zeta\eta^\nu$$

(definition of ζ, η). Now for any time t, define $\nu(t)$ to be either zero or the integer such that

$$\zeta\eta^{\nu(t)} \leqq t/\mid x_1 - x_2 \mid^2 < \zeta\eta^{\nu(t)+1}$$

if this integer exists. Then $t_{\nu(t)} \leqq t$ and $A(t) \leqq A(t_{\nu(t)}) = \sigma^{\nu(t)}$. Also,

$$\nu(t) \geqq (\log(t/\zeta \mid x_1 - x_2 \mid^2)/\log\eta) - 1.$$

From these observations, we conclude

$$\sigma^{\nu(t)} \leqq \sigma^{-1} \exp[(\log\sigma/\log\eta)\log(t/\zeta \mid x_1 - x_2 \mid^2)];$$

hence

$$A(t) \leqq \sigma^{-1}(t/\zeta \mid x_1 - x_2 \mid^2)^{\log\sigma/\log\eta},$$

or

$$\tfrac{1}{2}\int \mid T_1 - T_2 \mid dx \leqq \sigma^{-1}\zeta^{\alpha/2}(\mid x_1 - x_2 \mid/t^{\frac{1}{2}})^\alpha,$$

where $\frac{1}{2}\alpha = -\log\sigma/\log\eta$.

Both σ and η are determined by d. Specifically, $\sigma = 1 - \frac{1}{4}\phi(d)$ and $\eta = [\sigma^{-2}(1 + 4\mu/d)]^2$. An optimal choice of d in relation to $\phi(d)$ would maximize α. We may choose d arbitrarily as, $d^2 = c_1$, say; this will make α a function of μ and c_2/c_1 (proof omitted). In any case, even if we set $d = 1$, we obtain the estimate

$$(29) \qquad \int \mid S(x, t, x_1, t_0) - S(x, t, x_2, t_0) \mid \leqq A_1(\mid x_1 - x_2 \mid/(t - t_0)^{\frac{1}{2}})^\alpha,$$

where A_1 and α are a priori constants depending only on n, c_1 and c_2. Also, for the dual adjoint equation,

$$(30) \qquad \int \mid S(x_1, t, x_0, t_0) - S(x_2, t, x_0, t_0) \mid dx_0 \leqq A_1(\mid x_1 - x_2 \mid/(t - t_0)^{\frac{1}{2}})^\alpha.$$

With (30), we obtain the estimate for the continuity in space of a bounded solution of (1). If $T(x, t)$ satisfies (1) and $\mid T \mid \leqq B$ for $t \geqq t_0$, then

$$\mid T(x_1, t) - T(x_2, t) \mid \leqq \mid \int [S(x_1, t, x_0, t) - S(x_2, t, x_0, t_0)] T(x_0, t_0) dx_0 \mid$$
$$\leqq B \int \mid S(x_1, t, x_0, t_0) - S(x_2, t, x_0, t_0) \mid dx_0.$$

Hence,

$$(31) \qquad \mid T(x_1, t) - T(x_2, t) \mid \leqq BA_1(\mid x_1 - x_2 \mid/(t - t_0)^{\frac{1}{2}})^\alpha.$$

Part V: Time Continuity.

(31) gives half of (2); the remaining part, time continuity, can be derived from (31) and the moment bound (13). Let $T(x,t)$ be a solution of (1) with $|T| \leqq B$ for $t \geqq t_0$. Then for $t' > t > t_0$ we have

$$T(x,t) - T(x,t') = T(x,t) - \int S(x,t',\bar{x},t)T(\bar{x},t)\,d\bar{x}$$
$$= \int S(x,t',\bar{x},t)[T(x,t) - T(\bar{x},t)]\,d\bar{x},$$

since $\int S\,d\bar{x} = 1$. Therefore, $|T(x,t) - T(x,t')| \leqq$

$$\int S(x,t',\bar{x},t)\,|T(x,t) - T(\bar{x},t)|\,d\bar{x}$$
$$\leqq \int S(x,t',x+y,t)\,|T(x,t) - T(x+y,t)|\,dy.$$

Now we separate this integral into two parts, in terms of a radius ρ; one where $|y| \leqq \rho$ and one where $|y| > \rho$. Thus $|T(x,t) - T(x,t')| \leqq I_1 + I_2$, where

$$I_1 = \int_{|y|\leqq\rho} S(x,t',x+y,t)\,|T(x,t) - T(x+y,\,t)|\,dy \leqq BA_1(\rho/(t-t_0)^{\frac{1}{2}})^{\alpha}$$

(because $\int S\,dy = 1$), and

$$I_2 = \int_{|y|>\rho} S(x,t',x+y,t)\,|T(x,t) - T(x+y,t)|\,dy$$
$$\leqq 2B\rho^{-1}\int_{|y|>\rho} |y|\,S(x,t',x+y,t)\,dy \leqq 2B\mu(t'-t)^{\frac{1}{2}}/\rho.$$

Adding the two inequalities,

$$|T(x,t) - T(x,t')| \leqq BA_1(\rho/(t-t_0)^{\frac{1}{2}})^{\alpha} + 2B\mu(t'-t)^{\frac{1}{2}}/\rho,$$

and if we choose ρ so as to minimize the sum, then

$$\alpha A_1\rho^{1+\alpha} = 2\mu(t'-t)^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}\alpha},$$

and we obtain

$$|T(x,t) - T(x,t')| \leqq BA_2[(t'-t)/(t-t_0)]^{\frac{1}{2}\alpha/(1+\alpha)}, \tag{32}$$

where $A_2 = (1+\alpha)A_1(2\mu/\alpha A_1)^{\alpha/(1+\alpha)}$. This result (32), combined with (31) yields (2), with $A = \max(A_1, A_2)$.

Part VI: Elliptic Problems.

We treat elliptic problems as a special type of parabolic problem, one in which the coefficients of the equation are time independent and a time independent solution is sought. The Hölder continuity of solutions of uniformly elliptic equations of the form $\nabla \cdot (C \cdot \nabla T) = 0$ appears as a corollary of the result for the parabolic case. There may exist another proof of our result (3). P. R. Garabedian writes from London of a manuscript by Ennio de Giorgi containing such a result. See de Giorgi's note, reference [9].

Let $\mathcal{D}$ be a domain in space-time defined by the constraints $|x| \leqq \sigma$ and $t \geqq 0$. Then $\mathcal{D}$ is a solid semi-infinite spherical cylinder. Call $\mathcal{B}$ the points of the cylindrical surface or boundary of $\mathcal{D}$, where $|x| = \sigma$. Let $\mathcal{D}_0$ be the points of the base of $\mathcal{D}$, where $t = 0$. Define $\mathcal{B}^*$ as the total boundary of $\mathcal{D}$, the union $\mathcal{B} \cup \mathcal{D}_0$, of the base and cylindrical surfaces.

A "Dirichlet parabolic boundary value problem" is given when values of T are specified on $\mathcal{B}^*$ and we ask for a solution of (1) in $\mathcal{D}$ assuming these specified values on $\mathcal{B}$. The solution of the problem must depend linearly on the boundary values; also, the maximum and minimum principles must hold. These facts require that the solution $T(x, t)$ be determined in this way:

$$T(x,t) = \int T(\xi)\, d\rho(\xi; x, t). \tag{33}$$

Here (x, t) is a point of $\mathcal{D}$, ξ is any point of $\mathcal{B}^*$, and $d\rho(\xi; x, t)$ is a positive measure, associated with ξ, which has $\int d\rho = 1$ and which vanishes for $t(\xi) > t$. The time and space coordinates of the point ξ are called $t(\xi)$ and $x(\xi)$. We cannot pause here for a detailed justification of (33), but refer the reader to the literature.

We can define a boundary value problem for which we know the solution in advance by setting $T(\xi) = S(x(\xi), t(\xi), x_0, t_0)$ if $t_0 < 0$. Then the solution of the problem is $S(x, t, x_0, t_0)$, and from (33),

$$S(x,t,x_0,t_0) = \int S(x(\xi), t(\xi), x_0, t_0)\, d\rho(\xi; x, t). \tag{34}$$

This is a powerful identity; it enables us to convert information on fundamental solutions into information on $d\rho$, and in particular, we can obtain a moment bound for $d\rho$. Multiplying (34) by $|x - x_0|$ and integrating, we have

$$\int |x - x_0|\, S(x,t,x_0,t_0)\, dx_0 = \int\int |x - x_0|\, S(x(\xi), t(\xi), x_0, t_0)\, d\rho\, dx_0.$$

Hence

$$\mu(t-t_0)^{\frac{1}{2}} \geqq \int\int \{|x-x(\xi)| - |x_0-x(\xi)|\} S(x(\xi), t(\xi), x_0, t\ )\, d\rho\, dx_0,$$

so that

$$\mu(t-t_0)^{\frac{1}{2}} + \int\int |x_0-x(\xi)|\, S(x(\xi), t(\xi), x_0, t_0)\, dx_0 d\rho \geqq \int |x-x(\xi)|\, \int S(x(\xi), t(\xi), x_0, t_0)\, dx_0 d\rho.$$

Since $\int S\, dx_0 = 1$, and from the moment bound (13) again, we obtain

$$\mu(t-t_0)^{\frac{1}{2}} + \int \mu(t(\xi)-t_0)^{\frac{1}{2}} d\rho \geqq \int |x-x(\xi)|\, d\rho.$$

Now $d\rho$ vanishes unless $t(\xi) \leqq t$, and t_0 can be as near to zero as desired; also, $\int d\rho = 1$. Hence we can simplify the above to

$$2\mu t^{\frac{1}{2}} \geqq \int |x-x(\xi)|\, d\rho(\xi; x, t). \tag{35}$$

This moment bound (35) on $d\rho$ enables us to control the relative sizes of the effects of the two parts of the boundary in determining $T(x,t)$, where (x,t) is in $\mathcal{D}$. Thus

$$\int |x-x(\xi)|\, d\rho \geqq \int \big||x| - |x(\xi)|\big|\, d\rho \geqq (\sigma - |x|) \int_{\mathcal{B}} d\rho.$$

Hence

$$\int_{\mathcal{B}} d\rho(\xi; x, t) \leqq 2\mu t^{\frac{1}{2}}/(\sigma - |x|). \tag{36}$$

Now let $T(x)$ be a solution in a region $\mathcal{R}$ of n-space of $\nabla \cdot (C(x) \cdot \nabla T) = 0$, where $C(x)$ satisfies the uniform ellipticity condition with bounds c_1 and c_2. If we introduce time and define $T(x,t) = T(x)$, then $T(x,t)$ satisfies $\nabla \cdot (C \cdot \nabla T) = T_t$, which is of our form (1). Suppose x_1 and x_2 are two points of $\mathcal{R}$ and let $d(x_1, x_2)$ be the smaller of $d(x_1)$ and $d(x_2)$, the distances from the boundary of $\mathcal{R}$ of x_1 and x_2 (of course, $d(x_1, x_2)$ may be infinite). For any $\sigma < d(x_1, x_2)$, we can define $\mathcal{D}_1$ as the set of points (x,t) in space-time where $|x-x_1| \leqq \sigma$ and $t \geqq 0$; also, $\mathcal{D}_2$ can be defined for x_2, and the boundaries $\mathcal{B}_1$, $\mathcal{B}_2$, etc. can be defined in the obvious way. $T(x,t)$ can be regarded as a solution of a parabolic boundary value problem either in $\mathcal{D}_1$ or $\mathcal{D}_2$. Another problem with solution $T'(x,t)$ can be defined at first as an initial value problem in all space by setting $T'(x,0) = T(x)$ for all x where $\min(|x-x_1|, |x-x_2|) \leqq \sigma$, that is, $T'(x,t) = T(x)$ when $(x,t) \in \mathcal{D}_{10} \cup \mathcal{D}_{20}$, and setting $T'(x,0) = 0$ for all other x values. If $B(\sigma) = \max |T(x)|$ over the set of x values where $\min(|x-x_1|, |x-x_2|) \leqq \sigma$ then $|T'(x,0)| \leqq B(\sigma)$; furthermore, the solution $T'(x,t)$ satisfies $|T'|$

$\leqq B(\sigma)$ for all $t \geqq 0$ by the maximum principle. We can also regard $T'(x,t)$ as a solution of a boundary value problem, either in $\mathcal{D}_1$ or in $\mathcal{D}_2$, where the boundary values are just the values $T'(x(\xi),t(\xi))$ assumed there anyway.

By (33), for any $(x,t) \in \mathcal{D}_i$,

$$T(x,t) - T'(x,t) = \int [T(x(\xi),t(\xi)) - T'(x(\xi),t(\xi))]\, d\rho_i(\xi;x,t),$$

where $d\rho_i$ is the measure associated with $\mathcal{D}_i$, and $i = 1, 2$. Now $T(x,t) = T(x)$ is time independent, and on $\mathcal{D}_{i0}$ we have $T(x,t) = T'(x,t) = T(x)$. Therefore,

$$|T(x) - T'(x,t)| \leqq \int_{\mathcal{B}_i} |T(x(\xi)) - T'(x(\xi),t(\xi))|\, d\rho_i \leqq 2B(\sigma) \int_{\mathcal{B}_i} d\rho_i,$$

and

$$|T(x_i) - T((x_i,t)| \leqq 4B(\sigma)\mu t^{\frac{1}{2}}/\sigma,$$

by use of (36). With our Hölder continuity estimate (2) for solutions of $\nabla \cdot (C \cdot \nabla T) = T_t$ in free space, we can bound $|T'(x_1,t) - T'(x_2,t)|$. This, with the inequality above yields

$$|T(x_1) - T(x_2)| \leqq B(\sigma)A(|x_1 - x_2|/t^{\frac{1}{2}})^{\alpha} + 8\mu B(\sigma)t^{\frac{1}{2}}/\sigma,$$

valid for any positive t. Choice of the optimal t value gives an inequality of the form

$$|T(x_1) - T(x_2)| \leqq B(\sigma)A'(|x_1 - x_2|/\sigma)^{\alpha/(\alpha+1)}. \tag{37}$$

If $|T(x)| \leqq B$ in $\mathcal{R}$, we may set $\sigma = d(x_1,x_2)$ and obtain (3).

Appendix.

The methods used above can give more explicit results, such as an explicit lower bound for the Hölder exponent α. This takes the form $\alpha = \exp[-a_n(\mu^2/c_1)^{n+1}]$, where a_n depends only on the dimension n. However, a sharper estimate for α might take a quite different form. Numerical calculation of extremal examples might give a better picture.

The moment bound (13) serves to control the rate of dispersal of fundamental solutions. An iterative argument based on (33) and (35) obtains stronger results from (13). In this argument, a fundamental solution is treated as the solution of an array of parabolic boundary value problems, the boundaries being a sequence of spheres centered at the source of the fundamental solution. The result is as follows: let $\nu = [\rho/2\mu(t_2 - t_1)^{\frac{1}{2}}]$, the largest integer not greater than $\rho/2\mu(t_2 - t_1)^{\frac{1}{2}}$, then

$$\int_{|x_2-x_1|\geqq\rho} S(x_2,t_2,x_1,t_1)\,dx_2 \leqq (\pi/4)^{\nu/2}/(\nu/2)!$$
$$\leqq \exp[-\tfrac{1}{2}(\nu+1)\log(2(\nu+1)/\pi e)].$$

Hence,

$$(38)\qquad \int_{|x_2-x_1|\geqq\rho} S(x_2,t_2,x_1,t_1)\,dx_2$$
$$\leqq \exp\{-\rho\log(\rho/\pi e\mu(t_2-t_1)^{\frac{1}{2}})/4\mu(t_2-t_1)^{\frac{1}{2}}\}.$$

With (38), the reproductive identity (5), and the bound (7), we obtain a pointwise upper bound of the form

$$(39)\quad S(x_2,t_2,x_1,t_1)$$
$$\leqq k(t_2-t_1)^{-n/2}\exp[-k\,|\,x_1-x_2\,|\,(t_2-t_1)^{-\frac{1}{2}}\log(k\,|\,x_1-x_2\,|(t_2-t_1)^{-\frac{1}{2}})].$$

On the other hand, we obtain from (5) and (23) (or alternatively, from (38) and an analogue of (25)), by an argument resembling that which gave (25), the lower bound

$$(40)\qquad S(x_2,t_2,x_1,t_1) \geqq (t_2-t_1)^{-n/2}\phi^*(|\,x_1-x_2\,|/(t_2-t_1)^{\frac{1}{2}}),$$

where ϕ^* is an a priori function determined by c_1, c_2, and n. The inequality $S(x_2,t_2,x_1,t_1) \geqq P_aP_bP_c$, where

$$P_a = \min S(x_2,t_2,\bar{x},\tfrac{1}{2}(t_1+t_2)) \text{ for } |\,\bar{x}-x_1\,| \leqq \rho,$$
$$P_b = \min S(\bar{x},\tfrac{1}{2}(t_1+t_2),x_1,t_1) \text{ for } |\,\bar{x}-x_2\,| \leqq \rho,$$
$$P_c = \int d\bar{x}, \text{ where } |\,\bar{x}-x_1\,| \leqq \rho \text{ and } |\,\bar{x}-x_2\,| \leqq \rho,$$

can be used in a iterative argument to strengthen (40). For any $\epsilon > 0$, we obtain

$$(41)\quad S(x_2,t_2,x_1,t_1)$$
$$\geqq k_1(t_2-t_1)^{-n/2}\exp[-k_2(|\,x_1-x_2\,|/(t_2-t_1)^{\frac{1}{2}})^{2+\epsilon}],$$

where k_1 and k_2 depend on ϵ (and on c_1, c_2, and n).

With (38), (41) and (35), we can estimate the speed of convergence to assigned boundary values of the solution of an elliptic boundary value problem, provided the boundary is "tame" enough. A point ξ on the boundary $\mathcal{B}$ of a region $\mathcal{R}$ is called *regular* if there are two positive numbers ρ and ϵ such that any sphere with radius $\leqq\rho$ and centered at ξ has at least the fraction ϵ of its volume *not* within $\mathcal{B}$. Then there are constants D, σ, and β determined by ϵ, c_2/c_1, and n such that for any x in $\mathcal{R}$ with $|\,x-\xi\,| \leqq \sigma\rho$, we have

$$T(x) \geqq \min T(\bar{\xi}) - D\,|(x-\xi)/\rho\,|^{\beta},$$
$$T(x) \leqq \max T(\bar{\xi}) + D\,|(x-\xi)/\rho\,|^{\beta}, \text{ where } |\bar{\xi}-\xi| \leqq \rho \tag{42}$$

($\bar{\xi}$ represents a variable point on the boundary $\mathfrak{B}$).

From (42), it follows that the solution of an elliptic boundary value problem is continuous at the boundary if continuous values were assigned on the boundary and all boundary points are regular. With Hölder continuous boundary values, the solution is Hölder continuous in the region and at the boundary.

From the estimates above, we can fairly easily derive a "Harnack inequality" for parabolic equations:

$$T(x_2, t) \geqq F(T(x_1, t)/B, |x_1 - x_2|/(t-t_0)^{\frac{1}{2}}), \tag{43}$$

provided $0 \leqq T \leqq B$ for $t \geqq t_0$. F is an a priori function, determined by c_1, c_2 and n. For the elliptic case where T is non-negative in a sphere of radius r centered at the origin, the result takes the form

$$|\log(T(x')/T(x))| \leqq H(r[r - \max(|x|, |x'|)]^{-1}, |x-x'|/r). \tag{44}$$

The a priori function H is determined by c_2/c_1 and n. This result is less easily obtained than (43).

Parabolic or elliptic problems with Neumann boundary conditions can aparently be handled by a relatively straightforward rederivation of the estimates of this paper in the context of the Neumann boundary, obtaining ultimately the Hölder continuity of the solution for any typical boundary shape.

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THE ALGEBRAIC SIMPLICITY OF CERTAIN GROUPS OF HOMEOMORPHISMS.*

By R. D. ANDERSON.**

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1. Introduction. In this paper we show that certain groups of transformations of certain strongly setwise homogeneous spaces[1] are algebraically simple. For G a group of the type studied and for $A, B \in G$ with B not the identity, we shall show that A is the product of at most six conjugates[2] of B and B^{-1} (both B and B^{-1} possibly appearing in the same product). The theorem of section 4 asserts conditions under which certain groups can be seen to have this property and hence to be simple. The most interesting type of group to which the conditions are applicable seems to be the group of all homeomorphisms (or, where applicable, of all orientation—preserving homeomorphisms) of a sufficiently setwise homogeneous space. In particular, as corollaries of the theorem of section 4, we conclude that the groups of all homeomorphisms of the Cantor ternary set, the universal curve, the set of all rational numbers, and the set of all irrational numbers are simple. Also the groups of all orientation-preserving homeomorphisms of the universal plane curve, of S^2 (the 2-sphere) and of S^3 (the 3-sphere) are simple. The methods are not applicable to the group of all orientation-preserving homeomorphisms of S^1.[3] Some partial results on the group of all (orientation-preserving) homeomorphisms of S^n are noted at the end of section 3 and in

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[1] The space X under examination need not be homogeneous in the usual sense, i. e. for some $A, B \in X$ there need not be a homeomorphism of X onto itself carrying A onto B. But we shall require that there be a collection K of closed subsets of X such that for $k_1, k_2 \in K$, there is a homeomorphism of X onto itself carrying k_1 onto k_2. The precise nature of this "set-wise homogeneity" is given in the definitions to follow of K-structured and (G, K)-rotational.

[2] For $x, y \in G$, $x^{-1}yx$ or xyx^{-1} is called a conjugate of y.

[3] That the group of all orientation preserving homeomorphisms of S^1 is simple was shown by Schreier and Ulam in [6]. Recently Gordon Fisher has observed a modification of the method of argument of the present paper which makes it applicable to the S^1 case and to certain subgroups of the groups of all homeomorphisms of the line.

section 5. The theorem of section 5 asserts that most of the groups of our corollaries (at least) have the additional property that they contain uncountably many conjugate classes.

In 1947, Ulam and von Neumann [7] asserted that the group G of all orientation-preserving homeomorphisms of S^2 was simple and in fact that there exists an $N > 0$ such that if $A, B \in G$ with B not the identity, then A is the product of at most N conjugates of B. In a letter, Ulam has stated that N may be taken as 23. Note that the result of the present paper employs conjugates of both B and B^{-1} and hence does not include this part of the result of [7].

2. Definitions, notation and conventions. All spaces referred to are Hausdorff. Extra conditions of setwise homogeneity will be imposed. Let X denote a space and let G be a group of homeomorphisms of X onto itself. Let $G(X)$ be the group of all homeomorphisms of X onto itself. Let C denote the Cantor ternary set, M the universal curve,[4] N the universal plane curve,[5] Ra the rationals, and let Ir denote the irrationals (on the line).

Let e denote the identity of G, let G^0 be the set of all elements of G which are the identity inside some open subset of X. For $f \in G$ and $k \subset X$, $f \mid k$ denotes the homeomorphism f cut down to k. For $k \subset X$, $\mathrm{Cl}(k)$ denotes the closure of k, $F(k)$ denotes the boundary of k, $I(k)$ denotes $k - F(k)$, and $k^\sim$ denotes $X - k$. For $g \in G^0$, any set $k \subset X$ for which $k^\sim$ is non-null and $g \mid k^\sim$ is $e \mid k^\sim$ is said to *support* g or to be a *set of support* for g. For Y any collection of subsets of X, Y^* denotes the union of the elements of Y.

Let K (or $K(X)$ when convenient) be a collection of closed subsets of X. Then X is said to be *K-structured* if

(1) the elements of K are non-degenerate and are homeomorphic to each other,

(2) for $U \subset X$ with U open there exists $k \in K$ such that $k \subset U$,

(3) for $k \in K$, $\mathrm{Cl}(k^\sim) \in K$.

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[4] The universal curve was defined as a 1-dimensional analogue of the Cantor middle third set by punching holes out of a cube. It is characterized in [1] and [2] as a 1-dimensional locally connected continuum with no local cut points and with every open subset containing a 1-skeleton of a 4-simplex.

[5] The universal plane curve N is obtained by punching holes out of a 2-cell so as to get a 1-dimensional set with no local cut points. It is the usual "Swiss cheese" example and is not homogeneous. N contains points p and q (p on a bounding simple closed curve of a complementary domain in the plane and q not so) such that some simple closed curve containing p does not separate N while every simple closed curve containing q does separate N.

Agreement. *We henceforth assume K to exist and X to be K-structured.* Let $\hat{K}$ be the set of images of the elments of K under the various elements of G.

Remark 1. For C, Ra, and Ir, we shall let $K(C)$, $K(Ra)$, and $K(Ir)$ be the sets of all open and closed proper subsets of C, Ra, and Ir, respectively. For S^2 and S^3 we let $K(S^2)$ and $K(S^3)$ be the sets of all closed 2-cells and polyhedral 3-cells in S^2 and S^3 respectively (S^3 being regarded as triangulated). For M, we let $K(M)$ be the set of all closed subsets of M such that for $k \in K(M)$, (1) $I(k)$ and k^- are connected, (2) $\mathrm{Cl}[I(k)] \cap \mathrm{Cl}(k^-) = J(k)$ is homeomorphic to C, and (3) no connected subset of $\mathrm{Cl}[I(k)]$ or of $\mathrm{Cl}(k^-)$, open relative to $\mathrm{Cl}(I(k))$ or $\mathrm{Cl}(k^-)$ is disconnected by the omission of $J(k)$. For N, we let $K(N)$ be the collection of all closed sets each bounded by a simple closed curve in N which does not intersect the union of the set of boundaries of the complementary domains of N (as embedded in a plane).

We say that $K' \subset K$ is a *β-sequence* provided (1) K' is a countably infinite collection of disjoint sets with K'^* in the interior of some element of K and (2) $\mathrm{Cl}(K'^*) - K'^*$ is exactly one point $p \in X$ such that every neighborhood of p contains all but a finite number of the elements of K'.

Notation. We shall let $\{k_i\}$, $i = 0, \pm 1, \pm 2, \cdots$, denote a β-sequence and it shall always be assumed that a β-sequence is indexed over all positive, negative and zero integers.

We say that X is *(G,K)-rotational* (or has *(G,K)-rotationality*) provided that for any $k \in K$, there exist a β-sequence $\{k_i\}$ with $\cup k_i \subset k$ and elements $\rho_1, \rho_2 \in G^0$ supported on k such that

(1) $\rho_1(k_i) = k_{i+1}$ for each i,

(2) $\rho_2 \mid k_0$ is $\rho_1 \mid k_0$, $\rho_2 \mid k_{2i}$ is $\rho_1^{-2} \mid k_{2i}$ for $i > 0$, and $\rho_2 \mid k_{2i-1}$ is $\rho_1^2 \mid k_{2i-1}$ for $i > 0$,

(3) if, for each i, $\varphi_i \in G^0$ is supported on k_i, then there exists $\varphi \in G^0$ supported on $\cup k_i$ such that, for each i, $\varphi \mid k_i$ is $\varphi_i \mid k_i$, and

(4) for any $k' \in K$, there exists $\eta \in G$ such that $\eta(k') = k$.

We say that X is *(G,K)-homogeneous* provided that

(1) X is (G,K)-rotational,

(2) for any $k \in K$ and $g \in G$ for which $g(k) \cap k = \phi$ while $g(k) \cup k \neq X$, there exists $k' \in K$ and $\lambda \in G^0$, with λ supported on k', such that $\lambda \mid k$ is $g \mid k$, and

(3) for any $k_1, k_2, k_3, k_4 \in \hat{K}$ with $k_1 \cap k_2 = k_3 \cap k_4 = \phi$ and $k_1 \cup k_2 \neq X \neq k_3 \cup k_4$, there exists $\mu \in G$ such that $\mu(k_1) = k_3$ and $\mu(k_2) = k_4$.

Remark 2. It is easy to see (or is well known) that for each of C, Ra, Ir, and S^2 and the corresponding previously indicated groups, each space is (G, K)-homogeneous. It follows from results and techniques of [3] that for M and $G(M)$, M is also (G, K)-homogeneous. For G the group of all orientation-preserving homeomorphisms of S^3 and K the set of all polyhedral 3-cells in S^3 (under some triangulation), it follows from results of Bing [5] that S^3 is (G, K)-homogeneous. It follows from results and techniques of [4] and/or rather straightforward arguments that for G the group of all orientation-preserving homeomorphisms of N, N is (G, K)-homogeneous. (That there are orientation-preserving and non-orientation-preserving homeomorphisms of N also follows from [4]).

In Remark 4 of the next section we note that for any $n > 1$, S^n, the n-sphere, K a certain set of n-cells in S^n, and G the group of all (orientation-preserving) homeomorphisms of S^n, S^n is (G, K)-rotational.

Agreement. *In what follows we shall regard $X, K = K(X)$ and G as given with X K-structured and (G, K)-homogeneous unless weaker conditions are explicitly specified. We let h denote an element of G distinct from e. We assume other elements of G to be different from e where appropriate.*

We list some properties of G, K, and X which follow almost immediately from the definitions.

(A) Let m be a subset of X and $g \in G^0$ with g supported on m. Then for any $\psi \in G$, $\psi^{-1} g \psi$ is supported on $\psi^{-1}(m)$. If $\psi^{-1}(m)$ and m are disjoint, then $f = \psi^{-1} g^{-1} \psi g$ is supported on $m \cup \psi^{-1}(m)$ and $f \mid m$ is $g \mid m$, while $f \mid \psi^{-1}(m)$ is $\psi^{-1} g \psi \mid \psi^{-1}(m)$.

(B) Any conjugate of a product of conjugates of h and h^{-1} is itself a similar product of conjugates of h and h^{-1}; e.g. if

$$f = (g_1^{-1} h^{-1} g_1)(g_2^{-1} h g_2)(g_3^{-1} h^{-1} g_3),$$

then $\varphi^{-1} f \varphi = (\varphi^{-1} g_1^{-1} h^{-1} g_1 \varphi)(\varphi^{-1} g_2^{-1} h g_2 \varphi)(\varphi^{-1} g_3^{-1} h^{-1} g_3 \varphi)$.

(C) For $k_1, k_2 \in \hat{K}$ with $(k_1 \cup k_2)^- \neq \phi$, there exists a $k \in K$ such that $k_1 \cup k_2 \subset I(k)$. Note that as $(k_1 \cup k_2)^-$ is open, there exists a $k' \subset (k_1 \cup k_2)^-$, and $\mathrm{Cl}(k'^-) \in K$).

(D) X has no isolated points and each element of K is infinite.

(E) If $\alpha, \beta, \gamma \in G$ and m is any closed set in X, then in studying the action of $f = \alpha\beta\gamma$ on m we first look at γ on m, then at β on $\gamma(m)$ and finally at α on $\beta(\gamma(m))$.

3. Lemmas and theorems. In Theorem I and Lemma 1, we note consequences of (G, K)-rotationality of X (without requiring (G, K)-homogeneity).

THEOREM I. *Let X be (G, K)-rotational. For $g \in G^0$, g is the product of four conjugates of h and h^{-1} (appearing alternately).*

We first note that Theorem I follows from

LEMMA 1. *Let X be (G, K)-rotational. There exists $k_0 \in K$ such that for any $g_0 \in G^0$ with g_0 supported on k_0, g_0 is the product of four conjugates of h and h^{-1} (appearing alternately).*

For $g \in G^0$, g is supported on some element k of K. By condition (4) of (G, K)-rotationality, there exists an $\alpha \in G$ such that $\alpha(k)$ is k_0. Then $\alpha g \alpha^{-1} = g_0$ is supported on k_0 and $g = \alpha^{-1} g_0 \alpha$. Thus, by Note (B), as g_0 is the product of four conjugates of h and h^{-1} (alternately), so is g.

Proof of Lemma 1. Let $k \in K$ such that $k \cap [h(k) \cup h^{-1}(k)] = \phi$. That such a k exists follows from the existence of some p for which $h(p) \neq p$, property (2) of K-structured, and the continuity of h and h^{-1}. Let $\{k_i\}$ be a β-sequence with $\cup k_i \subset k$ and $\rho_1, \rho_2 \in G^0$ existing as in the definition of (G, K)-rotationality. The k_0 introduced here is to be the k_0 of the lemma. Let φ_i be $\rho_1{}^i g_0 \rho_1{}^{-i}$, $i \geqq 0$. By condition (3) of (G, K)-rotationality, there exists $\varphi \in G^0$ such that φ is supported on $\cup k_i$, $i \geqq 0$, $\varphi \mid k_0$ is $g_0 \mid k_0$, $\varphi \mid k_i$ is $e \mid k_i$, $i < 0$, and $\varphi \mid k_i$ is $\rho_1{}^i g_0 \rho_1{}^{-i} \mid k_i$ is $\varphi_i \mid k_i$ for $i > 0$.

Consider $f = h^{-1}\varphi^{-1} h \varphi$. By note A, f is supported on $[\cup k_i] \cup [\cup h^{-1}(k_i)]$ for $i \geqq 0$. Furthermore, $f \mid k$ is $\varphi \mid k$, whereas $f \mid h^{-1}(k)$ is $h^{-1}\varphi^{-1} h \mid h^{-1}(k)$. Let $\rho = (h^{-1}\rho_2 h)\rho_1{}^{-1}$. Then ρ is $\rho_1{}^{-1}$ over k and is $h^{-1}\rho_2 h$ over $h^{-1}(k)$.

Let

$$w = \rho^{-1} f^{-1} \rho f = \rho^{-1}\varphi^{-1} h^{-1} \varphi h \rho h^{-1} \varphi^{-1} h \varphi = (\rho^{-1}\varphi^{-1} h^{-1} \varphi \rho)(\rho^{-1} h \rho)(h^{-1})(\varphi^{-1} h \varphi).$$

Thus w is the product of four conjugates of h and h^{-1} (alternately).

We wish to check that w is g_0 and hence prove the lemma. To do so, we need merely track down the action of w on various subsets of X.

As f is supported on $Y = [\cup k_i] \cup [\cup h^{-1}(k_i)]$ for $i \geqq 0$, then clearly as $Y \subset \rho(Y)$, $w = \rho^{-1} f^{-1} \rho f$ is supported on Y. Each of f and ρ is the product of two elements of G^0, one supported on k and the other on $h^{-1}(k)$. Therefore,

we may consider the effect of w on k and $h^{-1}(k)$ separately in terms of those factors which are supported on k and $h^{-1}(k)$ respectively. Thus $w \mid k$ is $\rho_1\varphi^{-1}\rho_1^{-1}\varphi \mid k$ and $w \mid h^{-1}(k)$ is $(h^{-1}\rho_2^{-1}h)(h^{-1}\varphi h)(h^{-1}\rho_2 h)(h^{-1}\varphi^{-1}h) \mid h^{-1}(k)$.

We look first at $w \mid k$. As noted above, w is supported on $\cup k_i$, $i \geqq 0$. But $w \mid k_0$ is $\varphi \mid k_0$ is $g_0 \mid k_0$, for $\rho_1\varphi^{-1}\rho_1^{-1} \mid k_0$ is $e \mid k_0$. Now for $i > 0$, $w \mid k_i$ is $\rho_1\varphi^{-1}\rho_1^{-1}\varphi \mid k_i$ is

$$\rho_1(\rho_1^{i-1}g_0^{-1}\rho_1^{-i+1})\rho_1^{-1}(\rho_1^{i}g_0\rho_1^{-i}) \mid k_i \text{ is } \rho_1^{i}g_0^{-1}g_0\rho_1^{-i} \mid k_i$$

is $e \mid k_i$. We use property E here (and later). Thus $w \mid k$ is $g_0 \mid k$.

Finally, we look at $w \mid h^{-1}(k)$. By cancelling hh^{-1} where applicable, we note that $w \mid h^{-1}(k)$ is $h^{-1}\rho_2^{-1}\varphi\rho_2\varphi^{-1}h \mid h^{-1}(k)$, and to show that $w \mid h^{-1}(k)$ is $e \mid h^{-1}(k)$, we need merely show that $\rho_2^{-1}\varphi\rho_2\varphi^{-1} \mid k$ is $e \mid k$. Now $\varphi^{-1} \mid k_0 = g_0^{-1} \mid k_0$ and for $i > 0$, $\varphi^{-1} \mid k_i$ is $\rho_1^{i}g_0^{-1}\rho_1^{-i} \mid k_i$. Hence for $i > 0$, $\rho_2^{-1}\varphi\rho_2\varphi^{-1} \mid k_{2i}$ is $\rho_1^{2}(\rho_1^{2i-2}g_0\rho_1^{-2i+2})\rho_1^{-2}(\rho_1^{2i}g_0^{-1}\rho_1^{-2i}) \mid k_{2i} = e \mid k_{2i}$, $\rho_2^{-1}\varphi\rho_2\varphi^{-1} \mid k_{2i-1}$ is

$$\rho_1^{-2}(\rho_1^{2i+1}g_0\rho_1^{-2i-1})\rho_1^{2}(\rho_1^{2i-1}g_0^{-1}\rho_1^{-2i+1}) \mid k_{2i-1}$$

is $e \mid k_{2i-1}$, and for $i = 0$, $\rho_2^{-1}\varphi\rho_2\varphi^{-1} \mid k_0$ is $\rho_1^{-1}(\rho_1 g_0\rho_1^{-1})\rho_1 g_0^{-1} \mid k_0 = e \mid k_0$.

Thus Lemma 1 and Theorem I are established.

THEOREM II. *Let X be (G,K)-homogeneous. For any $k, k' \in K$ with $k \cap k' = \phi$ and $k \cup k' \neq X$ and for any $g \in G^0$ with g supported on k, there exists an $f \in G^0$ such that $f \mid k$ is $g \mid k$, $f \mid (k \cup k')^{\sim}$ is $e \mid (k \cup k')^{\sim}$ and f is the product of a conjugate of h and a conjugate of h^{-1}.*

We first note that Theorem II follows from

LEMMA 2. *Let X be (G,K)-homogeneous. For some $k_0, k_0' \in K$, $k_0 \cap k_0' = \phi$ and $k_0 \cup k_0' \neq X$, and for any $g_0 \in G$ with g_0 supported on k_0, there exists an $f_0 \in G^0$ such that $f_0 \mid k_0$ is $g_0 \mid k_0$, $f \mid (k_0 \cup k_0)^{\sim}$ is $e \mid (k_0 \cup k_0')^{\sim}$ and f is the product of a conjugate of h and a conjugate of h^{-1}.*

Assume Lemma 2. By condition (3) of (G,K)-homogeneity, there exists $\mu \in G$ such that $\mu(k_0) = k$ and $(k_0') = k'$. Let $g_0 \mid k_0$ be $\mu^{-1}g\mu \mid k_0$. Let f_0 be as in Lemma 2. Then $f = \mu f_0 \mu^{-1}$ satisfies the conditions of the theorem.

Proof of Lemma 2. Let $k_0 \in K$ such that $k_0 \cap [h(k_0) \cup h^{-1}(k_0)] = \phi$. Let $g_0 \in G_0$ with support on k_0. Then $f_0 = h^{-1}g_0^{-1}hg_0$ satisfies the conditions of the lemma with $k_0' = h^{-1}(k_0)$.

Remark 3. The order for the product of Theorem II could (by appropriate modification of the argument) be taken either as h, h^{-1} or as h^{-1}, h.

Remark 4. We suggest a construction which enables us to observe that if X is S^n, $n > 1$, and G is the group of all (orientation-preserving) homeomorphisms of S^n, there exists a K with respect to which X is K-structured and (G, K)-rotational.

We regard S^n as the one-point compactification of Euclidean n-space E^n. Let K be the set of (a) those closed n-cubes in E^n whose faces are parallel to the coordinate planes and (b) the closures of the complements of those n-cubes. Let $k \in K$ with center at the origin and edge of large size. Let now $\{k_i\}$ be a β-sequence with $\cup k_i \subset k$ such that k_0 has its center at $(1, 0, 0, \cdots, 0)$ whereas k_i, for $i < 0$, has its center at $(1/2^i, -1/2^i, 0, 0, \cdots, 0)$, k_{2i}, $i \geqq 0$, has its center at $(1/2^{2i}, 1/2^{2i}, 1/2^{2i}, 0, 0, \cdots, 0)$, while k_{2i-1}, $i > 0$, has its center at $(1/2^{2i-1}, 1/2^{2i-1}, -1/2^{2i-1}, 0, 0, \cdots, 0)$. Now it is clear that a ρ_1 can be defined as in the definition of (G, K)-rotational with the added provision that ρ_1 preserves geometric orientation and direction on each k_i. But then, similarly, it is clear that ρ_2 as in the definition can also be defined and hence S^n is (G, K)-rotational.

4. The main theorem and its corollaries.

THEOREM III. *Let X be (G, K)-homogeneous. Then for $g \in G$, g is the product of six conjugates of h and h^{-1} (written alternately or in the order $h, h^{-1}, h, h^{-1}, h, h^{-1}$) and hence G is simple.*

Proof. We use condition (2) of (G, K)-homogeneity (not previously employed). It is here that the orientation-preserving condition of certain groups becomes significant. Let $k \in K$ such that $g(k) \cap k = \phi$ and $g(k) \cup k \neq X$. Let $k' \in K$ for which $I(k') \supset k \cup g(k)$. Let $\alpha \in G^0$ supported on k' such that $\alpha \mid k$ is $g \mid k$. Then as $\alpha, (\alpha^{-1}g) \in G^0$ and $\alpha(\alpha^{-1}g) = g$, it follows from Theorem I that g can be written as the product of eight conjugates of h and h^{-1}. We sharpen this result slightly to get Theorem III. Let $k'' \cap k' = \phi$ and $k'' \cup k' \neq X$. By Theorem II, there is a $\beta \in G^0$ supported on k'' such that $\alpha\beta$ is the product of a conjugate of h and one of h^{-1} (in either order). Hence as $(\beta^{-1}\alpha^{-1}g) \in G^0$ and $(\alpha\beta)(\beta^{-1}\alpha^{-1}g) = g$, the theorem follows from Theorem I.

Noting Remarks 1 and 2, the following corollaries are immediate consequences of Theorem III.

COROLLARY 1. *If G is the group of all orientation-preserving homeomorphisms of N, S^2 or S^3 and $h \in G$ with $h \neq e$, then G is simple, and in fact, each element of G is the product of six conjugates of h and h^{-1}.*

COROLLARY 2. *If G is the group of all homeomorphisms of the Cantor set C, the universal curve M, the rationals Ra or the irrationals Ir and $h \in G$ with $h \neq e$, then G is simple, and in fact, each element of G is the product of six conjugates of h and h^{-1}.*

5. Conjugate classes. We give conditions on G sufficient to establish that G have uncountably many conjugate classes and we note that the groups of the corollaries of the last section as well as the group of orientation preserving homeomorphisms of S^n, $n > 1$, satisfy these conditions (except for the groups associated with M and N for which this argument is not designed).

Suppose that for some $k, k' \in K$ with $k' \subset k$ and for each $i > 1$, there exists a $g_i \in G$ such that

(1) g_i is supported on k,

(2) g_i is of period i on k', and

(3) g_i is not of period $j, j \neq 1, i$, on any homeomorph of k' different from k'. Then we say that G is *w-locally periodic.*

THEOREM IV. *If X is (G,K)-rotational and G is w-locally periodic, then G contains uncountably many distinct conjugate classes.*

Proof. Let $\hat{k} \in K$, let $\{k_i\}$ be a β-sequence with $\cup k_i \subset \hat{k}$ and let $\rho_1 \in G^0$ as in the definition of (G,K)-rotational such that ρ_1 is supported on $\hat{k}$ and for each i, $\rho_1(k_i) = k_{i+1}$. Let k and k' be as in the definition of w-locally periodic. Let $\sigma \in G$ such that $\sigma(k) = k_0$. Then for each i, $\sigma g_i \sigma^{-1}$ satisfies the conditions of periodicity above with respect to k_0 and $\sigma(k') = k_0'$. For $i > 0$, let $\varphi_i = \rho_1{}^i \sigma g_i \sigma^{-1} \rho_1{}^{-i}$. For each proper subsequence α of the positive primes let $\varphi_\alpha \in G$ which (1) is supported on $\cup k_i$, $i \in \alpha$, and (2) is φ_i over k_i for $i \in \alpha$. There are uncountably many (in fact, continuum-wise many) such sequences α. But for the distinct subsequences α and β of the positive primes, we show that φ_α and φ_β are not conjugate to each other. Without loss of generality, suppose φ_α is periodic with period p over some $k^0 \in K$ while φ_β is not periodic with period p over any homeomorphic image of an element of K. Suppose $\varphi_\alpha = f^{-1}\varphi_\beta f$; then $\varphi_\alpha{}^p$ is $(f^{-1}\varphi_\beta f)^p$ is $f^{-1}\varphi_\beta{}^p f$ which implies a contradiction.

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APPLICATIONS OF THE THEORY OF MORSE TO SYMMETRIC SPACES.*

By RAOUL BOTT[1] and HANS SAMELSON.[2]

To Marston Morse on his 65th birthday.

Introduction. In this sequel to [6], we bring the proof of results announced in [7] and [8]. Although [6] might be a good introduction, the present paper is relatively self contained.

Let Ω be the loop space on the connected compact Lie group K, and let S be the set of geodesics on K leading from a general point P to the identity e. (A point of K is general if it is contained in only one maximal torus of K.) To each $s \in S$ we assign an index (dimension), δ_s, equal to the number of (properly counted) singular points of K in the interior of s. The main result of [6] then states that $H_*(\Omega; \mathbf{Z})$ is *abstractly* isomorphic to S_*, the free module generated by the elements of S, and graded by the index δ. (Thus as a generator in S_*, the element $s \in S$ has dimension δ_s.)

In the present paper this result is extended in two ways: a) We define an explicit isomorphism $\gamma: S_* \to H_*(\Omega; \mathbf{Z})$; in other words, we describe a base of singular cycles for $H_*(\Omega; \mathbf{Z})$. b) We show that (at least with $\mathbf{Z}_2$ as coefficients) a completely analogous description of $H_*(\Omega)$ holds in several instances, notably when Ω is the loopspace of a symmetric space.

This program is carried out in chapters I and II. We start by constructing a homomorphism $\gamma: S_* \to H_*(\Omega)$ (at least mod 2) under rather general conditions: Ω is the loop space of a manifold M on which a compact Lie group K acts with at least one fixed point. The set S is now defined as above, with the fixed point playing the role of e and P being any point on a K-orbit of maximal dimension. The index $\delta_K(s)$ is computed again as before, the K-singular points of M being the points on lower-dimensional orbits. Using the K-singular points in s, we construct a manifold Γ_s (with $\dim \Gamma_s = \delta_K(s)$) and a homeomorphism $f_s: \Gamma_s \to \Omega$, for each $s \in S$. These sub-

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manifolds of Ω can be thought of as generalizations of the cycles in the loop space of a sphere which Morse constructed (cf. [16]). By assigning to each $s \in S$ the f_s-image of the fundamental cycle on Γ_s, we obtain the homomorphism $\gamma: S_* \to H_*(\Omega)$, in general mod 2, and over the integers if all the Γ_s are orientable. The details of this construction are given in Chapter I, nos. 1-5.

In general this homomorphism has no great virtue. However, and this is our main observation, if the action of K on M satisfies a certain infinitesimal condition, already introduced in [6] and there called variational completeness, then $\gamma: S_* \to H_*(\Omega)$ is an isomorphism onto. This is the content of Theorem I.

Intuitively, we like to think of variational completeness as absence of conjugate points on the decomposition space M/K. This is literally correct when K reduces to the identity, or $M \to M/K$ is a fibering. In the case of K acting on itself by inner automorphisms, this interpretation fits in with E. Cartan's description of this decomposition space as a Euclidean cell (with certain identifications on the boundary); the proof that variational completeness holds for this example was already given in [6].

The proof of Theorem I takes up the rest of Chapter I. The Morse theory enters vitally, and more delicately than in [6], where one merely applied the Morse inequalities.

In Chapter II, we prove that variational completeness holds in various instances (Theorem II). All of these cases are associated with symmetric spaces.

In Chapter III, we describe the ring $H^*(\Gamma_s; \mathbf{Z})$ for a generic s on the group K, in terms of the Cartan integers of K (Proposition 4.2). The rest of that chapter then is devoted to applications, of Theorem I and this description of $H^*(\Gamma_s)$, to the topology of Lie groups. The main results are outlined in no. 2 of that chapter. We have reproduced our computations with possibly greater detail than is currently the custom.

Chapter IV contains immediate applications of Theorems I and II to symmetric spaces in general. The results here are less complete than in Chapter III. The manifolds Γ_s need not be orientable so that we have to restrict ourselves to the coefficient domain $\mathbf{Z}_2$. We do not know whether the loopspace of a general symmetric space has ony two-torsion.

Chapter I. Action of a Group and the Theory of Morse.

1. Action of a group on a manifold. All manifolds considered in this paper are of class C^∞. They are not necessarily connected, but all components of a manifold are of the same dimension. If x is a point of the manifold M, then M_x denotes the tangent space of M at x. If $f: M \to M'$ is a map (always C^∞) of the manifold M into the manifold M', then $\dot{f}_x: M_x \to M'_{x'}$ denotes the differential of f at x, with $x' = f(x)$. If f is a curve in M, i.e. a map of the real line $E^1 = \boldsymbol{R}$ into M, we write $\dot{f}(t)$ for the tangent vector to f at t, that is, for the image under $\dot{f}_t$ of the standard tangent vector D_t (or $\frac{d}{dt}$) of E^1 at t.

Let K be a compact Lie group, and M a (paracompact) manifold of dimension r. *Let K act on M from the left,* i.e. *let there be given a map*

$$\pi: K \times M \to M \text{ of class } C^\infty \tag{1.1}$$

such that (with $\pi(k,x)$ abbreviated to $k \cdot x$) *the following conditions hold:*

$$e \cdot x = x, \text{ where } e \text{ is the identity of } K, \tag{1.2}$$

$$k \cdot (k' \cdot x) = (k \cdot k') \cdot x \text{ for } k, k' \in K, x \in M. \tag{1.3}$$

The K-orbit of a subset A is denoted by $K \cdot A$. Thus,

$$K \cdot A = \{k \cdot a : k \in K, a \in A\}. \tag{1.4}$$

The stabilizer (stability group) of A is denoted by K_A; this is the subgroup of K which keeps A pointwise fixed,

$$K_A = \{k \in K : k \cdot a = a, a \in A\}. \tag{1.5}$$

For each $x \in M$, the orbit $K \cdot x$ is a regular submanifold of M, C^∞-homeomorphic to the quotient space K/K_x.

It is also convenient to denote, for any $k \in K$, by $\pi_k: M \to M$ the map defined by $\pi_k(x) = k \cdot x$, and to abbreviate $\pi_k \circ f$ to $k \cdot f$, for any map f with values in M.

It is well known that M can be given a Riemannian metric that is invariant under the action of K, so that all the maps $\pi_k: M \to M$ become isometries. We assume that such a metric is assigned, and use the notation (X, Y) for the corresponding inner product of two vectors in M_x; the norm of X is written $|X|$.

The action of K on M induces a representation of the Lie algebra $\mathfrak{k}$ of

K by vector fields over M. These vector fields—the infinitesimal K-motions—can be defined in the following manner:

Let X be an element of $\mathfrak{k}$, and let $h: \boldsymbol{R} \to K$ be the corresponding 1-parameter subgroup, with $\dot{h}(0) = X$.

DEFINITION 1.6. *For $x \in M$, let $h_x: R \to M$ be the curve defined by $h_x(\alpha) = h(\alpha) \cdot x$ for any $\alpha \in \boldsymbol{R}$. The assignment $x \to \dot{h}_x(0)$ defines a vector field $\tilde{X}$ on M, which is called the infinitesimal K-motion corresponding to X.*

It is clear that the tangent space to the K-orbit $K \cdot x$ at x is given by the values of all possible infinitesimal K-motions at x:

$$(K \cdot x)_x = \{\tilde{X}(x) : X \in \mathfrak{k}\}. \tag{1.7}$$

The Riemannian metric in M defines in standard fashion the concept of distance (greatest lower bounds of length of curves connecting the two points), and the concept of geodesic. To be able to apply the theory of Morse, we shall assume that M is complete under its metric, so that geodesics can be continued indefinitely in both directions. A geodesic will then always be a non-degenerate C^∞-map of $\boldsymbol{R}$ into M, satisfying the usual differential equations for geodesics, parametrized proportionally to arc length. A geodesic segment s is the restriction of a geodesic g to an interval $[a, b]$ with $a < b$. The point $s(a)$ is the initial point, $s(b)$ is the terminal point of s. We write K_s for the stabilizer of $s([a, b])$; similarly for the stabilizer of a geodesic. The elements of K map geodesics into geodesics (the maps π_k are isometries).

2. K-transversality; the space $\Omega(M; \boldsymbol{R}, N)$. We introduce some basic definitions.

DEFINITION 2.1. *A geodesic g of M is called K-transversal (briefly: transversal) if for each $t \in \boldsymbol{R}$ the tangent vector $\dot{g}(t)$ is orthogonal to the K-orbit of the point $g(t)$ (i.e., orthogonal to the tangent space of the submanifold $K \cdot g(t)$ of M at $g(t)$); similarly for geodesic segments.*

There exist many transversal geodesics (unless K is transitive on M, in which case there are none at all!), as the next proposition shows.

PROPOSITION 2.2. *The geodesic g is transversal if there exists a $t_0 \in \boldsymbol{R}$ such that $\dot{g}(t_0)$ is orthogonal to $K \cdot g(t_0)$.*

In other words, transversality holds, provided it holds at a single point. The proof of Proposition 2.2 is an immediate consequence of Gauss' theorem on orthogonal trajectories of families of geodesics; we will give it in no. 6.

DEFINITION 2.3 (cf. [6], Def. 5.1). *If A, B are subsets of M, then $\Omega(M;A,B)$ denotes the metric space of all piecewise regular curves (maps u of the unit interval $[0,1]$) in M from A to B, i.e. with $u(0) \in A$, $u(1) \in B$, parametrized proportionally to arc length, with the distance between two curves u and u' defined as*

$$\sup_{t \in [0,1]} d(u(t),u'(t)) + |(L(u) - L(u')|;$$

here d is the distance in M, and L denotes the length of the curves.

Note that L is a continuous function on $\Omega(M;A,B)$.

Remark. The set $\Omega(M;A,B)$ is contained in the set $\Omega'(M;A,B)$ of all continuous curves (maps of $[0,1]$ into M) from A to B; if the latter, as customary, is topologized by the compact-open topology, then the inclusion map is continuous, and induces an isomorphism of the singular homology groups and also of the homotopy groups of the two spaces [17].

Let R be a point of M, and let N be the K-orbit of a point of M; to avoid trivial complications, assume $R \notin N$. The main object of study in this paper will be the space $\Omega = \Omega(M;R,N)$, to which we shall apply the theory of Morse. The next definition introduces a subset of Ω that is basic for this application.

DEFINITION 2.4. *$S = S(M;R,N)$ is the set of transversal geodesic segments, parametrized (proportionally to arc length) on $[0,1]$, from R to N, i.e. with initial point R and with terminal point on N.*

3. Fiber bundle concepts. We interpolate a section recalling some concepts from the theory of fiber bundles, specialized slightly to fit our purposes. By a principal bundle E for a compact Lie group G (the structure group) we mean a compact manifold on which G acts, from the right, without fixed points ($p \cdot g = p \Rightarrow g = e$). The decomposition space of E, derived from the decomposition of E into the orbits under G, is the base space B, a manifold. The natural map of E onto B is called the projection; it is differentiable; only the vectors tangent to the fibers are mapped into 0. One often writes E/G for B, particularly in the case where E is a group, G a subgroup and $B = E/G$ the space of left cosets of G.

We describe the standard construction of fiber spaces, associated to the principal bundle E for G: Suppose G operates, from the left, on another space F. The product of E and F over G, written $E \times_G F$, is obtained by identifying $(p \cdot g, x)$ with $(p, g \cdot x)$ in $E \times F$, or equivalently, by letting G

act on $E \times F$ via $(p,x)\cdot g = (p\cdot g, g^{-1}\cdot x)$ (which makes $E\times F$ into a principal G-bundle) and taking the base space. There exists a map of $E\times_G F$ onto B which makes the following diagram commutative:

$$\begin{array}{ccc} E\times F & \longrightarrow & E \\ \downarrow & & \downarrow \\ E\times_G F & \longrightarrow & B, \end{array} \tag{3.1}$$

where the other maps are the natural ones; this map is the induced fiber map with F as fiber.

Let another group G' act principally on F from the right, such that the actions of G and G' on F associate, and let G' act from the left on a third space F'. Then $E\times F\times F'$ is a principal $G\times G'$-bundle by the definition $(p,x,x')\cdot(g,g') = (p\cdot g, g^{-1}\cdot x\cdot g', g'^{-1}\cdot x')$. The base space, denoted by $E\times_G F\times_{G'} F'$, can in an obvious way be regarded also as $E\times_G (F\times_{G'} F')$ or $(E\times_G F)\times_{G'} F'$. Extension to more factors is immediate. Similarly, the two spaces $E\times_G F/G'$ and $(E\times_G F)/G'$ can be identified.

Let E, E' be two principal bundles with structure groups G, G' and base spaces B, B'. A map $\bar f: E\to E'$ is a bundle map or equivariant map relative to a homomorphism $f: G\to G'$, if $\bar f(p\cdot g) = \bar f(p)\cdot f(g)$ for $p\in E$, $g\in G$. Such a map induces a map $\bar f$ of B into B'. The spectral sequence of E (in homology) is then mapped into that of E'; the map has the obvious naturality properties with respect to E_2, and the transgression (cf. [18] for these concepts).

4. The K-cycle of a geodesic segment. Let $s:[0,1]\to M$ be an element of the set S (cf. (2.4)), i.e. a transversal geodesic segment with $s(0)=R$ and $s(1)\in N$. We shall attach to s a manifold Γ_s, the "K-cycle of s," and a homeomorphism f_s of Γ_s into Ω. We first describe Γ_s abstractly, and present its geometrical meaning in the next section.

DEFINITION 4.1. *The parameter value $t\in[0,1)$ (note that 1 is excluded) and the point $s(t)$ are called exceptional for s, if for the dimensions of the stabilizers $K_{s(t)}$ and K_s (cf. no. 1), the relation*

$$\dim K_{s(t)} > \dim K_s$$

holds.

Geometrically, the exceptional points are those where the segment encounters orbits of lower dimension. We shall show in (9.2) that each s has only a finite number of exceptional values, and that the exceptional values are related to the focal values.

Let $t_1 < t_2 < \cdots < t_n$ be the exceptional values of s; the stabilizers $K_{s(t_i)}$ are then defined, and we put, for brevity [as long as we consider a fixed s]:

$$K_i = K_{s(t_i)}, \qquad i = 1, \cdots, n. \tag{4.2}$$

The stabilizer K_s of the whole segment s is a subgroup of each K_i; it acts on K_i principally from the left and the right, by multiplication, and acts also from the left on the coset space K_n/K_s. Because of the importance of coset spaces with K_s as subgroup, we introduce the following notation:

(4.3) *If G is a group containing K_s, then G^s denotes the left coset space G/K_s.*

In the following fundamental definition, we make use of the notation introduced in no. 3.

DEFINITION 4.4. *The K-cycle of s, denoted by Γ_s, is the manifold*

$$K_1 \times_K K_2 \times_K \cdots \times_K K_{n-1} \times_K K_n{}^s;$$

note that the last factor, according to (4.3), *is K_n/K_s.*

The dimension of Γ_s is given by

$$\dim \Gamma_s = \sum_i (\dim K_i - \dim K_s). \tag{4.5}$$

If s has no exceptional values at all, we let Γ_s be a single point.

The manifold Γ_s can also be considered as the base space of $K_1 \times_{K_s} K_2 \times_{K_s} \cdots \times_{K_s} K_n$, under the right operation of K_s on K_n. A third description will be used below. Put

$$W_s = K_1 \times \cdots \times K_n, \tag{4.6}$$

and let the n-th power $(K_s)^n = K_s \times \cdots \times K_s$ operate on W_s by the rule

$$p \cdot q = (c_1 a_1, a_1^{-1} c_2 a_2, a_2^{-1} c_3 a_3, \cdots, a_{n-1}^{-1} c_n a_n) \tag{4.7}$$

for $p = (c_1, \cdots, c_n) \in W_s$, $q = (a_1, \cdots, a_n) \in (K_s)^n$.

One verifies that this is a principal bundle operation, and that the base space is exactly Γ_s. We denote by ψ_s the projection of W_s onto Γ_s. It should be noted that although $(K_s)^n$ is a subgroup of W_s, and although the orbit of the point $(e, \cdots, e)$ of W_s under the operation (4.7) is just $(K_s)^n$, the operation (4.7) is not identical with left (or right) group multiplication in W_s, and Γ_s is not homeomorphic with the coset space $W_s/(K_s)^n$; it is easily seen, however, that the two spaces have isomorphic homotopy groups π_i for $i > 1$, and that they agree in π_1 up to an extension.

For an example, let M be the 2-sphere $\{x: x_1^2 + x_2^2 + x_3^2 = 1\}$ in 3-space, and let K, isomorphic to the circle group S_1, be the group of rotations around the x_3-axis. Let $R = (1,0,0)$; $N = \{(0,0,1)\}$. Let s be the geodesic segment from $(1,0,0)$ to the southpole $(0,0,-1)$ to $(-1,0,0)$ to the northpole $(0,0,1)$. There is one exceptional value, $t = \frac{1}{3}$. The stabilizer K_1 is equal to K, the stabilizer K_s reduces to the identity. We have $\Gamma_s = W_s = K$, a circle.

5. Geometric form of the K-cycles. We come to the concrete representation of Γ_s as a submanifold of the path space Ω (cf. no. 2). Let again $t_1 < t_2 < \cdots < t_n$ be the exceptional t-values for s. Put $t_0 = 0$, $t_{n+1} = 1$, and denote by s_i the restriction of s to $[t_i, t_{i+1}]$, for $i = 0, \cdots, n$; the first one of these is possibly a single point, the others are non-degenerate geodesic segments. We write

$$s = s_0 + s_1 + \cdots + s_n \tag{5.1}$$

for this decomposition of s. With its help, we shall define a map $\tilde{f}_s$ of the manifold W_s (cf. (4.6)) into Ω. This map is then shown to be constant along the orbits of W_s under the action of $(K_s)^n$ (cf. (4.7)), and so defines the promised map f_s of the base space Γ_s into Ω.

For $(c_1, \cdots, c_n) = p \in W_s$, we put $\pi^k(p) = c_1 \cdot c_2 \cdot \cdots \cdot c_k$, for $k = 1, \cdots, n$, and $\pi^0(p) = e$.

DEFINITION 5.2. *$\tilde{f}_s(p)$ is the map of $[0,1]$ into M, whose restriction to $[t_i, t_{i+1}]$ equals $\pi^i(p) \cdot s_i$; i.e., for $t \in [t_i, t_{i+1}]$, one has $\tilde{f}_s(p)(t) = \pi^i(p) \cdot s_i(t)$.*

We write symbolically

$$\tilde{f}_s(p) = \pi^0(p) \cdot s_0 + \pi^1(p) \cdot s_1 + \cdots + \pi^n(p) \cdot s_n.$$

The map $\tilde{f}_s(p)$ is well defined: The value $\tilde{f}_s(p)(t)$ has a double definition for $t = t_{i+1}$, $0 \leqq i \leqq n-1$, namely $\pi^i(p) \cdot s_i(t_{i+1})$ and $\pi^{i+1}(p) \cdot s_{i+1}(t_{i+1})$. The two values are identical because firstly $s_i(t_{i+1})$ and $s_{i+1}(t_{i+1})$ are identical and secondly $\pi^{i+1}(p)$ differs from $\pi^i(p)$ by the factor c_{i+1} in the stabilizer of $s_{i+1}(t_{i+1})$.

Since K acts by isometries, each $\pi^i(p) \cdot s_i$ is isometric with s_i; it follows easily that the curve $\tilde{f}_s(p)$ is parametrized proportionally to arc length. The initial point of $\tilde{f}_s(p)$ is R (since $\pi^0(p) \cdot s_0(0) = e \cdot R = R$); the terminal point is $\pi^n(p) \cdot s_n(1) = c_1 \cdot \cdots \cdot c_n \cdot s(1)$ and so lies on N because $s(1)$ does. All this implies that the curve $\tilde{f}_s(p)$ is a point of $\Omega(M; R, N)$. It can be described as a polygon, consisting of first s_0, then s_1, moved by c_1, then s_2, moved by $c_1 \cdot c_2$, etc.

Let $\tilde{f}_s: W_s \to \Omega$ be the map which assigns to $p \in W_s$ the element $\tilde{f}_s(p)$ of Ω. The continuity of $\tilde{f}_s$ is clear. However, $\tilde{f}_s$ is in general not 1-1; in fact, each orbit under the action (4.7) of $(K_s)^n$ maps into a single point.

Let $q = (a_1, \cdots, a_n) \in (K_s)^n$. From the definition of the action of $(K_s)^n$ on W_s, one concludes that

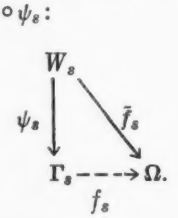
$$\pi^i(p \cdot q) = \pi^i(p) \cdot a_i. \tag{5.3}$$

Since $a_i \cdot s = s$, and so $a_i \cdot s_i = s_i$, it follows at once that

$$\tilde{f}_s(p \cdot q) = \tilde{f}_s(p).$$

This means that the map $\tilde{f}_s: W_s \to \Omega$ can be factored through the base space Γ_s under the projection ψ_s, i.e. that there exists a (unique) map $f_s: \Gamma_s \to \Omega$, such that $\tilde{f}_s = f_s \circ \psi_s$:

(5.4)



PROPOSITION 5.5. *f_s is a homeomorphism of Γ_s into Ω.*

Proof. Suppose $p = (c_1, \cdots, c_n)$ and $p' = (c'_1, \cdots, c'_n)$, two points of W_s, have the same $\tilde{f}_s$-image, i.e. the relations

$$\pi^i(p) \cdot s_i = \pi^i(p') \cdot s_i \text{ hold.}$$

Since none of the s_i for $1 \leqq i \leqq n$ is degenerate (so that $K_s = K_{s_i}$), it follows that $\pi^i(p)^{-1} \cdot \pi^i(p') = a_i$ belongs to K_s, for $1 \leqq i \leqq n$. This implies easily that $p' = p \cdot q$ with $q = (a_1, \cdots, a_n)$, and Proposition 5.5 is an immediate consequence.

Let $\omega_s \in \Gamma_s$ be the point $\psi_s(e, \cdots, e)$, i.e. the point whose inverse image under ψ_s is the fiber $(K_s)^n$; the f_s-image of ω_s is the geodesic segment s itself. Proposition 5.5 implies as a special case

PROPOSITION 5.5′. *The inverse image under f_s of the point s of Ω is the point ω_s of Γ_s.*

6. Jacobi fields; variational completeness. Our purpose is to formulate sufficient conditions for the cycles Γ_s to generate the homology of Ω. Before we can state our main result in this direction (cf. no. 7), we have to recall some facts concerning Jacobi fields. These vector fields can be defined in the following manner.

A geodesic variation $\{V_\alpha\}$ of a geodesic g in M is a C^∞-map $V: \boldsymbol{R} \times I \to M$, where I is an open interval containing 0, such that

(6.1) for each $\alpha \in I$ the map $V_\alpha: \boldsymbol{R} \to M$, defined by $V_\alpha(t) = V(t, \alpha)$, is a geodesic,

(6.2) $$V_0 = g.$$

DEFINITION 6.3. *The vector field η along g, defined by $\eta(t) = \partial V(t, 0)/\partial \alpha$, is called the Jacobi field, or J-field, determined by the geodesic variation V.*

It is well known that the J-fields along g form a vector space, which we shall call J_g, of dimension $2r$ (where $r = \dim M$), and that they can also be characterized as the solutions of the Jacobi differential equations (cf. [16]).

If h is a 1-parameter group in K, then the family $h(\alpha) \cdot g$, $\alpha \in \boldsymbol{R}$ is a geodesic variation of g; this implies that the restriction of any infinitesimal K-motion (cf. Definition 1.6) to g is a J-field along g. From the standard formula for the first variation of arc length, one derives easily that for any infinitesimal K-motion X the inner product $(\dot{g}(t), \tilde{X}(g(t))$ is independent of t.

We indicate now the proof of Proposition 2.2 as promised there. By (1.7), transversality of g at t means the vanishing of $(\dot{g}(t), \tilde{X}(g(t))$ for all $X \in \mathfrak{k}$. By the remark just made, this inner product will vanish for all t, provided it does so for any one particular value t_0.

We shall have to consider all those J-fields along g that vanish at some given parameter value t_0.

DEFINITION 6.4. *For any $t_0 \in \boldsymbol{R}$, $\Lambda_g(t_0)$ is the subspace of J_g consisting of those J-fields that vanish at t_0;*

$$\Lambda_g(t_0) = \{\eta \in J_g : \eta(t_0) = 0\}.$$

It can be shown that any J-field in $\Lambda_g(t_0)$ can be derived from a variation with fixed point at t_0, i.e., a variation $\{V_\alpha\}$ such that $V(t_0, \alpha) = g(t_0)$ for all $\alpha \in I$. Since a geodesic is determined by its tangent vector at any point, a variation with fixed point at t_0 is equivalent to a curve $Y(\alpha)$ in the tangent space $M_{g(t_0)}$, with $Y(0) = \dot{g}(t_0)$; in terms of V, we have $Y(\alpha) = \partial V(t_0, \alpha)/\partial t$. The corresponding J-field, derived by differentiation with respect to α, depends only on the vector $\dot{Y}(0)$. This amounts to a map θ from $M_{g(t_0)}$ (using the customary identification of a vector space with its tangent space at any of its points) into $\Lambda_g(t_0)$; actually, the map is an isomorphism of the two spaces involved. Suppose the variations are further restricted by the requirement that the tangent vectors of the V_α at t_0 lie in a given subspace W of $M_{g(t_0)}$;

then the associated J-fields will form a subspace W' of $\Lambda_{g(t_0)}$, and θ will map W onto W'. We omit the proofs of the various assertions made; they are classical, although usually stated in a different language.

DEFINITION 6.5. *Suppose g is a K-transversal geodesic (cf. no. 2); a J-field is called transversal if it is derived from a geodesic variation $\{V_\alpha\}$ of g in which all V_α are transversal geodesics.*

PROPOSITION 6.6. *If g is a transversal geodesic, then the restriction of any infinitesimal K-motion to g is a transversal J-field.*

This follows at once from the fact that the maps π_k (cf. no. 1) preserve transversality and that therefore each $h(\alpha)\cdot g$ is a transversal geodesic.

DEFINITION 6.7. *Let g be a transversal geodesic, and let t_0 be a real number. We denote by $J_g^\pi(t_0)$ the subspace of J_g of those transversal J-fields which at t_0 are tangent to the K-orbit of $g(t_0)$.*

This equals the J_g^N of [6], with N the orbit of $g(t_0)$. Remarks similar to those made after 6.4 apply. The dimension of $J_g^\pi(t_0)$ is $r = \dim M$ (to obtain $\eta \in J_g^\pi(t_0)$, one varies the vector $\dot{g}(t_0)$ in the r-dimensional manifold of vectors $(\neq 0)$ transversal to the orbit of $g(t_0)$).

Let $\tilde{X}$ be an infinitesimal K-motion, which therefore restricts to a transversal J-field η along the transversal geodesic g. It is clear from (1.7) that η is tangent to the orbit at every point of g, i.e. $\eta \in J_g^\pi(t)$ for all t.

The basic property which we require of our group K is laid down in the following definition (cf. [6], p. 261).

DEFINITON 6.8. *The action of K on M is variationally complete if every transversal J-field η, which is tangent to the K-orbits for two different points (parameter values) of the underlying transversal geodesic g (i.e., for which there exist $t_0, t_1 \in \mathbf{R}$, $t_0 \neq t_1$, such that $\eta \in J_g^\pi(t_0) \cap J_g^\pi(t_1)$) is induced by K, i.e. is the restriction to g of an infinitesimal K-motion.*

We get an equivalent definition, (6.8′), if instead of requiring tangency at two points, we require tangency at one point and vanishing at another point, i.e. $\eta \in J_g^\pi(t_0) \cap \Lambda_g(t_1)$. Clearly, if (6.8) holds, then also (6.8′) holds. We consider the reverse implication: Let (6.8′) hold, and let η be a transversal J-field in $J_g^\pi(t_0) \cap J_g^\pi(t_1)$. Since $\eta(t_0)$ is tangent to the orbit, there exists an infinitesimal K-motion $\tilde{X}$, such that $\tilde{X}(g(t_0)) = \eta(t_0)$. Denote by ζ the restriction of $\tilde{X}$ to g. Then $\eta - \zeta$ is a transversal J-field which vanishes at t_0 and is tangent to the orbit at t_1. By (6.8′), it is therefore induced by K; but then $\eta = (\eta - \zeta) + \zeta$ is also induced by K.

7. The main theorem. For any space X and Abelian group $\boldsymbol{G}$, we denote by $H_r(X;\boldsymbol{G})$ the r-th singular homology group of X with coefficients in $\boldsymbol{G}$; similarly, $H_r(X,A;\boldsymbol{G})$ denotes the relative homology group of X modulo the subset A. We put $H_*(X;\boldsymbol{G}) = \sum_r H_r(X;\boldsymbol{G})$. We denote the integers by $\boldsymbol{Z}$ and the integers modulo 2 by $\boldsymbol{Z}_2$. In the case $\boldsymbol{G} = \boldsymbol{Z}$, we write simply $H_*(X)$, etc. If $f: X \to Y$ is a map, then f_* is the induced homology map.

Cohomology is indicated by upper indices, $H^r(X;\boldsymbol{G})$, etc. If $\boldsymbol{G}$ is a commutative ring with unit, then $H^*(X;\boldsymbol{G})$ is a graded $\boldsymbol{G}$-algebra (cup product). We write $x(y)$ for the value of a cocycle x on a cycle y.

A fundamental cycle on a manifold, with coefficients in $\boldsymbol{Z}$ or $\boldsymbol{Z}_2$, is the sum of fundamental cycles (i.e. generators of the homology groups in the highest dimension) for the various components of the manifold.

DEFINITION 7.1 (cf. [6], Proposition 6.1). *The defect $\delta(Q)$ of a point Q of M is the difference between the maximum of the dimensions of K-orbits in M and the dimension of the K-orbit of Q.*

Let R be a point with maximal orbit dimension, i.e. with $\delta(R) = 0$; it is clear that $\delta(Q)$ can be expressed in terms of stabilizers by

$$\delta(Q) = \dim K_Q - \dim K_R. \tag{7.2}$$

For a geodesic segment $s: [a,b] \to M$ we define the defect δ_s by

$$\delta_s = \sum_{a \leqq t < b} \delta(s(t)); \tag{7.3}$$

we shall show later (after Proposition 9.2) that δ_s is finite if $\delta(s(a)) = 0$.

For each s in the set $S = S(M;R,N)$ (cf. no. 2), we have constructed in no. 4, 5 the manifold Γ_s and the map f_s of Γ_s into Ω. Note that Γ_s is not necessarily connected. For each such s, let $\bar{\gamma}_s$ denote the (unique) fundamental cycle over $\boldsymbol{Z}_2$ of Γ_s. We call S orientable (or better, K-orientable) if all Γ_s are orientable manifolds [i.e., all components of each Γ_s orientable]. In this case, an orientation of S is a function γ which to each $s \in S$ assigns a fundamental cycle γ_s over $\boldsymbol{Z}$ of Γ_s. Only the orientation of the principal component, containing the point $\psi_s(e,\cdots,e)$, will actually matter later. Suppose R lies on an orbit of maximal dimension. It is well known that the stabilizer of any point Q on s, sufficiently close to R, is contained in K_R since s is orthogonal to the orbit of R (existence of a slice [15]). The maximality condition on R implies that the e-components of K_Q and K_R, and therefore also those of K_s and K_R, coincide. We conclude now from (4.5), (7.2), (7.3):

If R has defect 0, then for any $s \in S(M; R, N)$, the relation

$$\dim \Gamma_s = \delta_s \text{ holds.} \tag{7.4}$$

Let $\bar{S}_*$, resp. S_*, denote the $\mathbf{Z}_2$-vector space, resp. the free abelian group, with the elements of S as basis, graded by assigning to any s the value δ_s as dimension.

We can now state our first main theorem.

THEOREM I. *Let the action of K on M be variationally complete and let R lie on an orbit of maximal dimension. Then the elements $f_{s*}(\bar{\gamma}_s)$, for $s \in S$, form a basis of (the $\mathbf{Z}_2$-vector space) $H_*(\Omega; \mathbf{Z}_2)$. If S is K-orientable, then $H_*(\Omega; \mathbf{Z})$ is free abelian, with the elements $f_{s*}(\gamma_s)$, for $s \in S$ and γ any orientation of S, as a basis.*

In other words: The assignment $s \to f_{s*}(\bar{\gamma}_s)$, resp. $s \to f_{s*}(\gamma_s)$ in the K-orientable case, induces a gradation preserving isomorphism $H_*(\Omega; \mathbf{Z}_2) \approx \bar{S}_*$, resp. $H_*(\Omega; \mathbf{Z}) \approx S_*$ in the K-orientable case.

The two versions of the theorem are equivalent because of (7.4).

The proof of Theorem I, based on the theory of Morse, will be given in the following sections.

We note that Theorem I constitutes an improvement over the results of [6] since it contains a construction of the homology of Ω, whereas Proposition 6.1 of [6], together with the Morse inequalities, essentially gives a majorisation of the Betti numbers of Ω.

Remarks. (a) Consider the case where K reduces to the identity. Each Γ_s is then a point. Variational completeness becomes simply absence of conjugate points. Theorem I says that Ω has no homology in positive dimensions, and that the cardinality of S is equal to the cardinality of the set of arc-components of Ω, i.e. equal to the order of the fundamental group $\pi_1(M)$ of M. If M is simply connected, this means that any two points are connected by exactly one geodesic.

(b) The action of S_1 on S_2, described in no. 2, is easily seen to be variationally complete. The element $f_{s*}(\gamma_s)$, for the s described there, with $R = (1, 0, 0)$, $N = \{(0, 0, 1)\}$, is the first Morse cycle, of dimension 1, in $\Omega(S_2)$; an alternative description is as follows: to each point x on the equator of S_2, associate the path consisting of the great circle arc from $(1, 0, 0)$ to the southpole plus the meridian through x (from $(0, 0, -1)$ to $(0, 0, 1)$); the image of the equator under this map into Ω is the Morse cycle in question. The constructions of Theorem I were motivated by this example.

8. Review of the theory of Morse. We recall some facts and definitions concerning the theory of Morse (cf. [16], [17]).

The values of the length function L on the subset S (made up of the transversal geodesics) of the function space Ω are the stationary values of L. Assume that, for any real l, the collection of elements of S, whose length is $\leqq l$, is finite. The stationary values, in natural order, form then a strictly increasing finite or infinite sequence $\{c_i\}$, $i = 1, 2, \cdots$; if the sequence is infinite, it diverges. If the sequence is finite, one extends it arbitrarily to a strictly increasing, diverging one. To each c_i corresponds then only a finite number of elements of S. Let Ω_i be the subset of Ω determined by $\Omega_i = \{u \in \Omega : L(u) \leqq c_i\}$, the set *up to* level c_i, and let $\Omega_i^- = \{u \in \Omega : L(u) < c_i\}$, the set *strictly below* level c_i; let Ω_0 and Ω_0^- be the empty set.

We define a group E_1 as the direct sum over i of the relative homology groups $H_*(\Omega_i, \Omega_{i-1}; \boldsymbol{G})$, and similarly E_1^- as the direct sum of the groups $H_*(\Omega_i, \Omega_i^-; \boldsymbol{G})$; the notation E_1, E_1^- is used because of the connection with spectral sequences. The following two propositions express Morse's evaluation of the group E_1.

PROPOSITION 8.1. *The inclusions* $(\Omega_i, \Omega_{i-1}) \subset (\Omega_i, \Omega_i^-)$ *induce isomorphisms of* $H_*(\Omega_i, \Omega_{i-1}; \boldsymbol{G})$ *and* $H_*(\Omega_i, \Omega_i^-; \boldsymbol{G})$, *and induce therefore an isomorphism of* E_1 *and* E_1^-.

We sketch briefly how this proposition follows from Morse's work. Put $w_i = \{s \in S : L(s) = c_i\}$, and $\Omega(c) = \{u \in \Omega : L(u) \leqq c\}$ for $c \geqq 0$. The main construction (cf. [17, Satz, p. 66]) consists in finding, for any c with $c_i \leqq c < c_{i+1}$, a deformation D_t, $0 \leqq t \leqq 1$, of the pair $(\Omega(c), \Omega_i^- \cup w_i)$ such that always $D_t(\Omega(c)) \subset \Omega(c)$, $D_t(\Omega_i^- \cup w_i) \subset \Omega_i^- \cup w_i$, and moreover $D_1(\Omega(c)) \subset \Omega_i^- \cup w_i$. It follows that the relative homology groups $H_*(\Omega(c), \Omega_i^- \cup w_i)$ vanish. This implies (by the exact sequence of a triple) that $H_*(\Omega(c), \Omega_i)$ vanishes; one concludes easily that also $H_*(\Omega_{i+1}^-, \Omega_i)$ vanishes, and 8.1 follows. (We note that part of the Theorem on p. 275 in [6] is stated incorrectly; the map mentioned there should be replaced by the two inclusions

$$(\Omega_i^- \cup w_i, \Omega_i^-) \subset (\Omega(l_i), \Omega_i^-) \text{ and } (\Omega(l_i), \Omega(l_{i-1})) \subset (\Omega(l_i), \Omega_i^-).)$$

For non-compact, complete manifolds the argument has to be modified slightly.

To each geodesic segment s in S is assigned an index λ_s (reative to the manifold N): Recall that s is parametrized on $[0, 1]$; call g the underlying geodesic. A value $t_0 \in [0, 1)$ and the point $s(t_0)$ are called focal (for N) if there exists a non-trivial transversal Jacobi field along g that is tangent

to N for $t=1$ and vanishes for $t=t_0$, i.e., if the space $J_g^\pi(1) \cap \Lambda_g(t_0)$ is of positive dimension. The index $\lambda_s(t_0)$ of t_0 is the number of linearly independent such fields: $\lambda_s(t_0) = \dim J_g^\pi(1) \cap \Lambda_g(t_0)$. It is shown in [16] that each s has only a finite number of focal values.

DEFINITION 8.2. *The index λ_s (relative to N) of the geodesic segment s is given by*

$$\lambda_s = \sum_{0 \leq t < 1} \lambda_s(t) = \sum_{0 \leq t < 1} \dim J_g^\pi(1) \cap \Lambda_g(t).$$

The segment s is called non-degenerate (in the sense of Morse) if $\lambda_s(0) = 0$, i.e. if 0 is not a focal value. If all $s \in S$ are non-degenerate, then the finiteness assumption, made in the third sentence of this section, is satified. This is an immediate consequence of the implicit function theorem; for another approach cf. [16], Ch. VII, Theorem 11.1.

Assuming s non-degenerate, then there exists, according to Morse, a Euclidean simplex σ_s, of dimension λ_s, and a map $\phi_s : \sigma_s \to \Omega$, satisfying the following conditions: (a) the barycenter of σ_s goes into the point s of Ω, (b) the function $L \circ \phi_s$, induced by the length function on Ω, takes its maximum at the barycenter of σ_s, and only there, (c) $L \circ \phi_s$ is of class C^∞ near the barycenter, and the maximum is non-degenerate (i.e. the quadratic terms of the Taylor expansion constitute a negative definite form). Any singular simplex satisfying (a), (b), (c) will be said to be associated to s or "hanging over s" (using an intuitive term motivated by thinking of L, the length, as indicating the "height" of the points of Ω).

Let $\dot{\sigma}_s$ denote the boundary of σ_s. Then ϕ_s maps the pair $(\sigma_s, \dot{\sigma}_s)$ into the pair (Ω_i, Ω_i^-), with i determined by $c_i = L(s)$, by condition (b). Recall that the relative homology group $H_r(\sigma_s, \dot{\sigma}_s; \boldsymbol{G})$ is non-trivial only for $r = \dim \sigma_s = \lambda_s$.

PROPOSITION 8.3. *Assume that all $s \in S$ are nondegenerate; choose for each s an associated singular simplex (σ_s, ϕ_s). Let ξ_s, resp. $\bar{\xi}_s$, denote a generator of $H_{\lambda_s}(\sigma_s, \dot{\sigma}_s; \boldsymbol{G})$ for $\boldsymbol{G} = \boldsymbol{Z}$, resp. $\boldsymbol{Z}_2$. Then, for each i, the group $H_*(\Omega_i, \Omega_i^-)$ is free abelian, with a basis consisting of the $\phi_{s\,*}$-images of the elements ξ_s, for all s with $L(s) = c_i$; similarly, the ϕ_{s*}-images of the elements $\bar{\xi}_s$, for $L(s) = c_i$, form a basis of the $\boldsymbol{Z}_2$-vector space $H_*(\Omega_i, \Omega_i^-; \boldsymbol{Z}_2)$.*

We note that this implies $H_*(\Omega_i, \Omega_i^-; \boldsymbol{G}) = 0$ if there is no s with $L(s) = c_i$; this will happen, under our convention, for large i, if the number of stationary values is finite.

The simplest situation in which one can deduce the homology of Ω from E_1 is that in which the condition of "completability" holds:

CONDITION 8.4. *For each i and r the map $j_i\colon H_r(\Omega_i; \boldsymbol{G}) \to H_r(\Omega_i, \Omega_i^-; \boldsymbol{G})$, induced by the inclusion of (Ω_i, Ω_0) in (Ω_i, Ω_i^-) is onto.*

Geometrically this means that every relative cycle of $\Omega_i \bmod \Omega_i^-$, represented by an associated simplex "hanging over" a point s of $\Omega_i - \Omega_i^-$, can be completed "below level $c_i = L(s)$" to an absolute cycle of Ω.

PROPOSITION 8.5. (A) *Suppose all $s \in S$ are non-degenerate and the completability condition* (8.4) *holds for $\boldsymbol{G} = \boldsymbol{Z}$. For each $s \in S$, let x_s be an element of $H_q(\Omega_i)$, with i determined by $c_i = L(s)$, such that $j_i(x_s) = \phi_{s*}(\xi_s)$. Let z_s be the image of x_s in $H_*(\Omega)$ under the inclusion $\Omega_i \subset \Omega$. Then $H_*(\Omega)$ is a free abelian group, and the set $\{z_s\}$, $s \in S$, is a basis.*

(B) *A similar statement holds for the case $\boldsymbol{G} = \boldsymbol{Z}_2$, with elements $\bar{x}_s \in H_*(\Omega_i)$, and with the set $\{\bar{z}_s\}$, $s \in S$, forming a base of the $\boldsymbol{Z}_2$-vector space $H_*(\Omega; \boldsymbol{Z}_2)$.*

Note that the z_s and $\bar{z}_s$ are completions of the relative cycles $\phi_{s*}(\xi_s)$ and $\phi_{s*}(\xi_s)$.

We indicate briefly how Proposition 8.5 follows from (8.1) and (8.3) under condition 8.4, for the case of $\boldsymbol{G} = \boldsymbol{Z}$. From (8.1) and (8.3), one concludes that each $H_*(\Omega_i, \Omega_{i-1})$ is free. From the exactness of the homology sequence of (Ω_i, Ω_{i-1}) and (8.4), one concludes that the sequence

$$0 \to H_*(\Omega_{i-1}) \to H_*(\Omega_i) \to H_*(\Omega_i, \Omega_{i-1}) \to 0$$

is exact. One proves now inductively that each $H_*(\Omega_i)$ is free, and imbedded as direct summand in $H_*(\Omega_{i+1})$ under the inclusion $\Omega_i \subset \Omega_{i+1}$. Since $H_*(\Omega)$ is easily seen to be the direct limit of the $H_*(\Omega_i)$, it follows that it is free abelian. The statement concerning the basis $\{z_s\}$ for $H_*(\Omega)$ is verified by tracing the ξ_s through the construction.

9. Proof of Theorem I. Beginning. We shall show that the hypotheses of Proposition 8.5 are satisfied, if those of Theorem I are.

PROPOSITION 9.1. *Let the point R lie on a K-orbit of maximal dimension and suppose the action of K is variationally complete; then all geodesic segments making up the set $S = S(M; R, N)$ are non-degenate in the sense of Morse.*

Proof. Let s be an element of S, carried by the geodesic g. We have to show that the index $\lambda_s(0)$, which equals the dimension of $J_g^\pi(1) \cap \Lambda_g(0)$, vanishes. By the variational completeness assumption 6.8, any J-field in

$J_g^\pi(1) \cap \Lambda_g(0)$ is induced by an infinitesimal K-motion (cf. (1.5)) which in turn is derived from a variation of the form $h(\alpha) \cdot g$, where h is a 1-parameter group in K with the property: $h(\alpha) \cdot R = R$ for all α (the variation must leave R fixed, since the J-field vanishes at R). In other words, the elements $h(\alpha)$ are contained in the stabilizer K_R. As in the proof of (7.4), one sees that the e-components of K_g and K_R coincide. This means that the 1-parameter group h above leaves every point of g fixed, and the associated J-field vanishes identically, q.e.d.—The next step is to prove that the completability condition 8.4 is satisfied, and that in fact the K-cycles Γ_s under the maps f_s can be used as completions. The proof is more complicated than that of Proposition 9.1. Let again s be an element of S.

PROPOSITION 9.2. *For any* $t \in [0,1)$, *the inequality* $\dim K_{s(t)} - \dim K_s \leqq \lambda_s(t)$ *holds; if the action of* K *is variationally complete, then equality obtains.*

Proof. Every X in the Lie algebra $\mathfrak{k}$ produces, through the corresponding infinitesimal K-motion $\tilde{X}$, a transversal Jacobi field; because of (1.7), this field belongs automatically to $J_g^\pi(1)$. The field $\tilde{X}$ vanishes at the point $s(t)$ exactly if X belongs to the Lie algebra $\mathfrak{k}_{s(t)}$ of $K_{s(t)}$. This means that there is a map, obviously linear, of $\mathfrak{k}_{s(t)}$ into $J_g^\pi(1) \cap \Lambda_g(t)$. The kernel of this map consists of those X whose associated $\tilde{X}$ vanishes identically along s; it is clear that this is just the Lie algebra $\mathfrak{k}_s$ of K_s. The inequality

$$\dim \mathfrak{k}_{s(t)} - \dim \mathfrak{k}_s \leqq \dim J_g^\pi(1) \cap \Lambda_g(t)$$

of (9.2) is an immediate consequence. Assume now variational completeness. This means that every J-field in $J_g^\pi(1) \cap \Lambda_g(t)$ is induced by an infinitesimal K-motion. The map of $\mathfrak{k}_{s(t)}$ into $J_g^\pi(1) \cap \Lambda_g(0)$ introduced above is onto, and the equality of (9.2) follows.

We note that Proposition 9.2 implies the statements about exceptional values made after Definition 4.1 and after (7.3).

PROPOSITION 9.3. *If the action of* K *is variationally complete, then for every* $s \in S$, *the dimension of the* K*-cycle* Γ_s *equals the index* λ_s.

Proof. By 4.5, we have $\dim \Gamma_s = \sum_i (\dim K_{s(t_i)} - \dim K_s)$, where the sum runs over the exceptional values of s. Proposition 9.3 follows immediately from the Definition 8.2 of λ_s and (9.2). We note that (9.3) and (7.4) imply

(9.4) *If the action of* K *is variationally complete and* $\delta(R) = 0$, *then* $\dim \Gamma_s = \lambda_s = \delta_s$ *for every* $s \in S$.

10. A deformation of the K-cycles. Let s be a geodesic of S, and let i be the number of its stationary level: $L(s) = c_i$. In (5.4), we constructed a map $f_s: \Gamma_s \to \Omega$, and it is clear from the construction that all polygons making up the image set of f_s have the same length. Hence f_s maps Γ_s into Ω_i.

PROPOSITION 10.1. *Under the hypotheses of Theorem* I, *there is only one geodesic segment in the set $f_s(\Gamma_s)$, namely s itself; all other elements are geodesic polygons with at least one corner.*

Proof. Since 0 now is not an exceptional value by (9.1), it follows that the interval s_0 on s, introduced in (5.1), is non-degenerate. All elements of $f_s(\Gamma_s)$ are geodesic polygons beginning with s_0. If two geodesic segments have a non-degenerate segment in common, they are identical, q.e.d.

It is possible, and basic for the application of the theory of Morse, to deform f_s in such a way that all $f_s(x)$, except s itself, are replaced by shorter curves, by "cutting across the corners" of the polygons. Actually, a sharper statement has to be made.

PROPOSITION 10.2. *If the segment s is non-degenerate in the sense of Morse, then there exists a homotopy $f_s^u: \Gamma_s \to \Omega$, $0 \leqq u \leqq 1$, such that*

(a) $f_s^0 = f_s$,

(b) $f_s^u(\omega_s) = s$ *for all* $u \in [0,1]$,

(c) f_s^u *maps* $\Gamma_s - \omega_s$ *into* Ω_i^-, *whenever* $0 < u \leqq 1$,

(d) *for any* $u \in (0,1]$, *the induced length function* $L_s^u = L \circ f_s^u$ *on* Γ_s, *which by* (b), (c) *takes it maximum,* $L(s)$, *exactly at* ω_s, *has a non-degenerate critical point at* ω_s.

Assertion (d) means that L_s^u is differentiable near ω_s, and that, in terms of a local coordinate system $\{y_i\}$ at ω_s, the Hessian of L_s^u, i.e. the matrix $(\partial^2 L_s^u / \partial Y_i \partial Y_j \,|_{\omega_s})$, is non-singular; it will then automatically be negative definite, since L_s^u has a maximum at ω_s.

We first define a homotopy $\bar{f}_s^u$ of the map $\bar{f}_s$ of W_s into Ω (cf. no. 5).

Let U be an open neighborhood of s in M, invariant under K and with compact closure; such sets exist. It is known from Riemannian geometry that there is a positive $a = a(U)$ such that any closed sphere V of radius a with center in $\bar{U}$ is geodesically convex (so that any two points of V are joined within V by a unique geodesic segment that varies smoothly with the points and whose length equals the distance of the points). Let ϵ be a number such that

$$(10.3) \quad 0 < \epsilon < a(U),$$

(10.3′) $2\epsilon < L(s_i)$, $i = 0, \cdots, n$, where the s_i are the intervals on s, defined by the exceptional points (cf. (5.1)).

For any $u \in [0,1]$, we let s_i^u be the following restrictions of s, with $u' = u \cdot \epsilon / L(s)$:

$$
\begin{aligned}
&s_0^u = s \mid [0, t_1 - u'], \\
&s_i^u = s \mid [t_i + u', t_{i+1} - u'], \text{ for } i = 1, \cdots, n-1, \\
&s_n^u = s \mid [t_n + u', 1].
\end{aligned} \tag{10.4}
$$

Because of (10.3′), the indicated intervals of $[0,1]$ are all non-degenerate:

$$2u' \leqq 2\epsilon / L(s) < L(s_i)/L(s) = t_{i+1} - t_i.$$

We denote by $\alpha_i(u)$, resp. $\beta_i(u)$, the initial, resp. terminal point of the segment s_i^u.

Let $p = (c_1, \cdots, c_n)$ be any point of W_s. Using the notation introduced in no. 5, we let $r_i^u(p)$ be the unique shortest geodesic segment from $\pi^i(p) \cdot \beta_i(u)$ to $\pi^{i+1}(p) \cdot \alpha_{i+1}(u)$, parametrized from $t_{i+1} - u'$ to $t_{i+1} + u'$, for $i = 0, \cdots, n-1$.

The existence of $r_i^u(p)$ comes from the fact that the two points which it is supposed to connect lie with the ϵ-sphere around the point $\pi^i(p) \cdot s_i(t_{i+1}) \in U$, and (10.3) applies.

DEFINITION 10.5. *$\tilde{f}_s^u(p)$ is the geodesic polygon, parametrized on $[0,1]$ proportionally to arc length, made up of the segments*

$$\pi^0(p) \cdot s_0^u, \quad r_0^u(p), \quad \pi^1(p) \cdot s_1^u, \quad r_1^u(p), \cdots, \quad r_{n-1}^u(p), \quad \pi^n(p) \cdot s_n^u.$$

As indicated before, each $\tilde{f}_s^u(p)$ is a geodesic polygon in Ω (i.e. beginning at R, and ending transversally on N), obtained from $\tilde{f}_s(p)$ by "cutting across the corners." The construction is pictured in fig. 1. It follows from local minimum properties of geodesics in convex spheres that $\tilde{f}_s^u(p)$, for $u > 0$, is actually shorter than $\tilde{f}_s(p)$, provided there is at least one corner on $\tilde{f}_s(p)$.

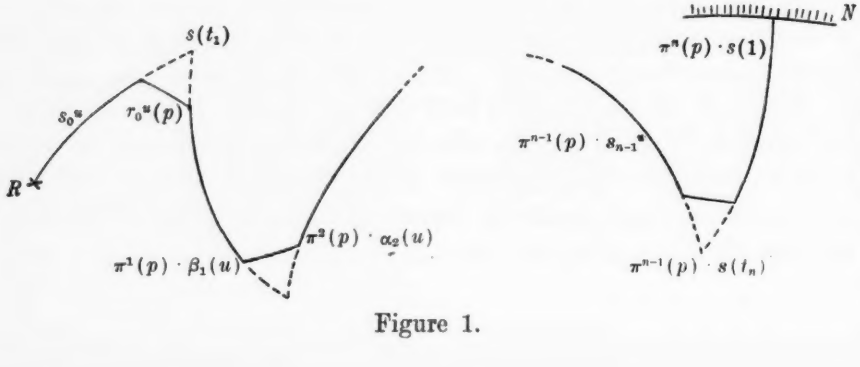


Figure 1.

11. The critical point on Γ_s. It is clear that with the choice of ϵ in (10.3), (10.3′), the point $\tilde{f}_s{}^u(p)$ of Ω depends continuously on the pair (p,u). Since $\tilde{f}_s{}^u(p)$ is derived from the polygon $\tilde{f}_s(p)$ by a geometrical construction, it follows at once that two points p, p' of W_s which are equivalent under the action 4.6 of $(K_s)^n$ on W_s, will give rise to the same modified polygon; i.e. $p' = p \cdot q$ implies $\tilde{f}_s{}^u(p') = \tilde{f}_s{}^u(p)$. This means that the map $\tilde{f}_s{}^u$ of W_s into Ω can be factored through Γ_s, with an induced map $f_s{}^u\colon \Gamma_s \to \Omega$. Clearly $f_s{}^u$ is a homotopy with properties (a), (b) of Proposition 10.2. The induced length function $L_s{}^u$ is clearly constant for $u = 0$. Since 0 is not exceptional for s, Proposition 10.1 applies. By the last remark of no. 10, the function $L_s{}^u$, for $u > 0$, will take a value less than $L(s)$ at every point x of Γ_s, different from ω_s; this proves (10.2 (c)).

The proof of (10.2 (d)) is more complicated. We note first that each $L_s{}^u$ is of class C^∞, since all the points involved in the construction of the geodesic polygons are C^∞-functions of the parameters occurring. We prove (d) in the following equivalent form.

PROPOSITION 11.1. *Let s be non-degenerate in the sense of Morse [and suppose $\dim \Gamma_s > 0$ to rule out a trivial case]; let x be a C^∞-curve in Γ_s, defined on some interval $(-a,a)$ with $x(0) = \omega_s$, and $\dot{x}(0) \neq 0$; put $F = L_s{}^u \circ x$ with any u in $(0,1]$; then the second derivative of F at the origin 0 is negative: $F''(0) < 0$.*

It is clear from what has been proved already that $F(0) = L(s)$ and $F'(0) = 0$, since F has a maximum at 0. For the proof of (11.1), we take a vector Y at $(e,\cdots,e)$ in W_s that projects into $\dot{x}(0)$ under the map $\psi_s\colon W_s \to \Gamma_s$, and let y be a curve in W_s, through $(e,\cdots,e)$ with $\dot{y}(0) = Y$, that projects into x, so that $\psi_s(y(v)) = x(v)$. Both Y and y exist since ψ_s is a fiber map. We have then $F(v) = L \circ \tilde{f}_s{}^u(y(v))$, since $f_s{}^u$ is induced by $\tilde{f}_s{}^u$.

Suppose first that y is of the form $y(v) = (e,\cdots,e,y_{i_0}(v),e,\cdots,e)$, i.e. that it has only one non-constant component. This means that the geodesic polygon $\tilde{f}_s(y(v))$, for any v, has exactly one corner at parameter value t_{i_0}.

Let $\phi(v)$ stand for the length of the geodesic segment from $s(t_{i_0} - u') = \beta_{i_0-1}(u)$ to $y_{i_0}(v) \cdot s(t_{i_0} + u') = y_{i_0}(v) \cdot \alpha_{i_0}(u)$, which cuts across the corner of $\tilde{f}_s(y(v))$ to form $\tilde{f}_s{}^u(y(v))$. We clearly have $F(v) = L(s) - 2u \cdot \epsilon + \phi(v)$. The vector Y is not in the kernel of ψ_s, and is therefore not tangent to the submanifold $\psi_s{}^{-1}(\omega_s) = (K_s)^n$ of W_s. This means that $\dot{y}_{i_0}(0)$ is not tangent

to K_s. Since $t_{i_0}+u'$ is not exceptional for s, this implies that the curve defined by $z(v)=y_{i_0}(v)\cdot s(t_{i_0}+u')$ has a non-zero tangent vector at $v=0$. The assertion $\phi''(0)<0$ reduces therefore to the following: Let a, b be the end points of a geodesic segment of length 2ϵ, whose midpoint c lies in $\bar{U}$; let $b(v)$ be a C^∞-curve such that (1) $b(0)=b$, (2) $b(v)$ lies on the ϵ-sphere around c, (3) $\dot{b}(0)\neq 0$; then for $d(v)$ defined as the distance from a to $b(v)$, the relation $d''(0)<0$ holds. This property of geodesic spheres (elementary for Euclidean space) is well known to hold, provided $a(U)$ is chosen sufficinetly small.

We consider now an arbitrary curve y, without the restriction made above. The vector $Y=(\dot{y}_1(0),\cdots,\dot{y}_n(0))$ is again not tangent to $(K_s)^n$ at $(e,\cdots,e)$, so that at least one of the $\dot{y}_i(0)$, say $\dot{y}_{i_0}$, *is* not tangent to the submanifold K_s of K_{i_0} at e. We put $y_0(v)=(e,\cdots,e,y_{i_0}(v),e,\cdots,e)$, and compare $\tilde{f}_s{}^u(y(v))$ with $\tilde{f}_s{}^u(y_0(v))$. The shortening process at the corner at t_{i_0} reduces both curves by the same amount from the original value $L(s)$. The latter curve has only this one corner, the former may have several corners; it follows that the former is not longer than the latter. Putting $F_0(v)=L\circ\tilde{f}_s{}^u(y_0(v))$, we have therefore $F(v)\leqq F_0(v)$. The reasoning above applies to F_0, since y_0 has only one non-constant component. We have therefore $F_0''(0)<0$. In addition, we have $F(0)=F_0(0)=L(s)$, and $F'(0)=F_0'(0)=0$, as noted before. It is elementary that then $F''(0)\leqq F_0''(0)$, and so finally $F''(0)<0$. Propositions 11.1 and 10.2 are proved.

12. Proof of Theorem I completed. For each $s\in S$, choose a Euclidean simplex σ_s, with $\dim\sigma_s=\dim\Gamma_s$, and a regular C^∞-homeomorphism ρ_s of σ_s into Γ_s such that ρ_s sends the barycenter of σ_s into ω_s. Define $\phi_s\colon\sigma_s\to\Omega$ by $\phi_s=f_s{}^u\circ\rho_s$ for some positive u. By (9.3) and (10.2), the singular simplex (σ_s,ϕ_s) is associated to s in the sense of no. 8. Put $g_s=f_s{}^u$, considered as a map into Ω_i.

$$\begin{array}{ccccc}
 & & \Gamma_s & \xrightarrow{\quad g_s \quad} & \Omega_i \\
 & & \Big\downarrow{\scriptstyle j_s} & & \Big\downarrow{\scriptstyle j_i} \\
(\sigma_s,\dot{\sigma}_s) & \xrightarrow{\rho_s} & (\Gamma_s,\Gamma_s-\{\omega_s\}) & \xrightarrow{g_s} & (\Omega_i,\Omega_i^-)
\end{array}$$

We consider the commutative diagram above, where the vertical maps are simply inclusion maps. Let first $\mathbf{Z}_2$ be taken as coefficient group. It is clear by excision that the j_s-image of the fundamental cycle of Γ_s equals the ρ_s-image of the fundamental cycle of $(\sigma_s,\dot{\sigma}_s)$; with the notations of Theorem I and

(8.3) we have $j_{s*}(\bar{\gamma}_s) = \rho_{s*}(\bar{\xi}_s)$. Applying g_{s*} (with $g_{s*} \circ \rho_{s*} = \phi_{s*}$) and using the commutativity of the diagram, one concludes that $\phi_{s*}(\bar{\xi}_s) = j_{i*}(g_{s*}(\bar{\gamma}_s))$, so that $\phi_{s*}(\bar{\xi}_s)$ is contained in $j_{i*}(H_{\lambda_s}(\Omega_i; \boldsymbol{Z}_2))$. Since this holds for every s, it follows now from (8.3) that the completability condition 8.4 holds for $G = Z_2$.

For the case of $\boldsymbol{G = Z}$, assume that S is K-orientable, and let γ be an orientation of S, as defined prior to Theorem I. Again it is clear, by excision, that for a suitably chosen generator ξ_s of $H_{\lambda_s}(\sigma_s, \dot{\sigma}_s)$ the relation $j_{s*}(\gamma_s) = \rho_{s*}(\xi_s)$ holds. The rest of the argument goes as before, and shows that 8.4 holds.

Non-degeneracy of all s has been proved in (9.1) already. With non-degeneracy and completability following from the hypotheses of Theorem I, we can now apply Proposition 8.5 with $\bar{x}_s = g_{s*}(\bar{\gamma}_s)$, or, in the K-orientable case, with $x_s = g_s(\gamma_s)$. Theorem I follows directly if we remark that the image of $g_{s*}(\bar{\gamma}_s)$ under the inclusion $\Omega_i \subset \Omega$ is $f_{s*}{}^u(\bar{\gamma}_s)$, and that $f_{s*}{}^u$ is identical with f_{s*}, by the homotopy axiom; similarly for γ_s.

Chapter II. Variational Completeness in Symmetric Spaces.

1. Action of $K \times K$ and of K. Let G be a compact connected Lie group, and let K be a closed connected subgroup, which then itself is also a Lie group. The direct product $K \times K$ acts from the left on G under the definition

$$(a, b) \cdot c = a \cdot c \cdot b^{-1} \tag{1.1}$$

with $a, b \in K$, $c \in G$; this is an action in the sense of Chapter I, no. 1. We introduce in G a Riemannian metric invariant under left and right translations. It is well known that such metrics exist, and that the geodesics in such a metric are the 1-parameter subgroups and their cosets. In general, we shall follow the notation of [6]. But we write $a \cdot X$ or aX instead $l_a(X)$ for the left translate of the vector X by the element a of G. The adjoint action of G on its Lie algebra $\mathfrak{g}$ is then given by $Ada \cdot x = a \cdot X \cdot a^{-1}$. For the Lie product $[\ ,\ ]$ and the invariant inner product or Killing form $(\ ,\ )$ in $\mathfrak{g}$ we have then the usual relations:

$$\begin{aligned} &\text{(a)} \quad \mathrm{Ad}a \cdot [X, Y] = [\mathrm{Ad}a \cdot X, \mathrm{Ad}a \cdot X], \\ &\text{(b)} \quad (X, Y) = (\mathrm{Ad}a \cdot X, \mathrm{Ad}a \cdot X), \\ &\text{(c)} \quad ([X, Y], Z) + (Y, [X, Z]) = 0, \\ &\text{(d)} \quad d\,\mathrm{Ad}e^{tY} \cdot X/dt\,|_{t=0} = [Y, X] = \mathrm{ad}Y \cdot X. \end{aligned} \tag{1.2}$$

As usual, G/K denotes the homogeneous space formed by the left cosets xK, $x \in G$, under the decomposition topology, with the induced C^∞-manifold structure; the group G acts on the left on G/K by $y \cdot (xK) = (yx)K$. Let p denote the natural map of G on G/K. The Riemannian metric of G induces a Riemannian metric on G/K, invariant under the action of G: If X' is a vector at the point $x' \in G/K$, let X be a vector at any point x in $p^{-1}(x')$, orthogonal to the manifold xK and with $\dot{p}(X) = X'$; define the norm $|X'|$ of X' to be equal to the norm $|X|$ of X; the definition is independent of the choice of x involved. In particular, let $\mathfrak{p}$ denote the subspace of the Lie algebra $\mathfrak{g}$ orthogonal to the Lie algebra $\mathfrak{k}$; then $\dot{p}$ gives an isometry of $\mathfrak{p}$ with the tangent space of G/K at $p(e) = e'$. The group K acts on G/K, as subgroup of G. Since K is the stability group of e', it acts on the tangent space $(G/K)_{e'}$. It also acts on $\mathfrak{g}$ by the adjoint transformations $\mathrm{Ad}k$. The subspaces $\mathfrak{k}$ and $\mathfrak{p}$ of $\mathfrak{g}$ are invariant under $\mathrm{Ad}K$, and the action of K on $\mathfrak{p}$ so defined is equivalent, via the map $\dot{p}$, to the action of K on $(G/K)_{e'}$.

2. Symmetric spaces and variational completeness. We recall E. Cartan's concept of symmetric space [9].

DEFINITION 2.1. *The pair (G,K) is called a symmetric pair, if there exists an involution $*$ (i.e. automorphism of order two), of G, written $x \to x^*$, such that K is the e-component of the fixed group of the involution (i.e. of the group $\{x : x^* = x\}$).*

If (G,K) is a symmetric pair, then G/K is a symmetric space; actually, one can divide G by any closed subgroup of G between K and the fixed group of the involution.

In Chapter I, (8.4), we have defined variational completeness which forms the hypothesis of Theorem I. The following theorem provides cases were this hypothesis holds.

THEOREM II. *Let (G,K) be a symmetric pair; then the action of $K \times K$ on G, the action of K on G/K and the action of K on $\mathfrak{p}$, as defined in no. 1, are variationally complete.*

The proof of this theorem is a generalization of that of Proposition 7.1 in [6].

We begin with the action of $K \times K$ on G. Theorem II, for this case, takes the form of Proposition 2.2 below; note that the role of K in Chapter I is now taken by $K \times K$. (For the definitions of transversal geodesic, transversal Jacobi field and infinitesimal $K \times K$-motion see Chapter I, (2.1), (8.2) and no. 1.)

PROPOSITION 2.2. *Let g be a $K \times K$-transversal geodesic of G; let η be a transversal Jacobi field along g that vanishes for $t = 0$ and is tangent to the $K \times K$-orbit of $g(1)$ for $t = 1$; then η is induced by an infinitesimal $K \times K$-motion.*

For the proof, we shall describe, in Lemma 4.1, the space of all transversal J-fields vanishing for $t = 0$; in Lemma 5.1, we describe the space of all infinitesimal $K \times K$-motions; Proposition 2.2 will then follow from a comparison of the two spaces.

3. Tangent and transversal space of an orbit. Basic for the proof are the following relations in the Lie algebra $\mathfrak{g}$ of G, which are well known to be equivalent to the existence (locally) of the involution which defines our symmetric pair (G, K):

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}; \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}; \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{3.1}$$

We also recall

$$\mathfrak{p} = \mathfrak{k}^{\perp} \text{ (orthogonal complement in } \mathfrak{g}). \tag{3.2}$$

Next we describe the tangent space of an orbit and its orthogonal complement.

DEFINITION 3.3. *For any $a \in G$, we put*

$$\mathfrak{k}^a = \mathfrak{k} + \mathrm{Ad}a^{-1} \cdot \mathfrak{k}; \quad \mathfrak{k}_a = \mathfrak{k} \cap \mathrm{Ad}a^{-1} \cdot \mathfrak{k}$$

$$\mathfrak{p}_a = \mathfrak{p} \cap \mathrm{Ad}a^{-1} \cdot \mathfrak{p}.$$

The relations (3.1) imply similar relations for the spaces defined in (3.3); we give the ones that we shall need later.

$$[\mathfrak{k}_a, \mathfrak{p}_a] \subset \mathfrak{p}_a; \quad [\mathfrak{p}_a, \mathfrak{p}_a] \subset \mathfrak{k}_a; \tag{3.4}$$

$$\mathfrak{p}_a = (\mathfrak{k}^a)^{\perp}. \tag{3.5}$$

(3.4) follows easily from (3.1) and the fact that $\mathrm{Ad}a^{-1}$ is an automorphism of $\mathfrak{g}$. Similarly, (3.5) follows from (3.2) and the fact that $\mathrm{Ad}a^{-1}$ is an isometry of $\mathfrak{g}$.

PROPOSITION 3.6. *The spaces $\mathfrak{k}^a$ and $\mathfrak{p}_a$ are the left translates, under a^{-1}, of the tangent space and the transversal space to the orbit $K \cdot a \cdot K$ at the point a.*

Proof. The tangent space to $K \cdot a \cdot K$ at a is clearly $\mathfrak{k} \cdot a + a \cdot \mathfrak{k}$. Applying

a^{-1} on the left, one gets $\mathfrak{k}^a$. The assertion for $\mathfrak{p}_a$ follows from (3.5) and the isometric character of left translation.

The space $\mathfrak{p}_a$ admits a decomposition relative to any of its elements, which we derive now.

DEFINITION 3.7. *For any $X \in \mathfrak{g}$, the centralizer $\mathfrak{g}_X$ (a subalgebra of $\mathfrak{g}$) consists of the elements $Z \in \mathfrak{g}$ with $[X,Z]=0$.*

LEMMA 3.8. *For any $X \in \mathfrak{p}_a$, the space $\mathfrak{p}_a$ admits the orthogonal direct sum decomposition*

$$\mathfrak{p}_a = \mathrm{ad}X(\mathfrak{k}_a) \oplus (\mathfrak{p}_a \cap \mathfrak{g}_X).$$

Proof. By (3.4), $\mathrm{ad}X(\mathfrak{k}_a)$ is a subspace of $\mathfrak{p}_a$. Take Z in $\mathfrak{p}_a$, orthogonal to this subspace. Then, for any $Y \in \mathfrak{k}_a$, we have $0 = ([X,Y],Z) = -(Y,[X,Z])$. Since $[X,Z]$ belongs to $\mathfrak{k}_a$ by (3.4) and since Y is arbitrary in $\mathfrak{k}_a$, we have $[X,Z]=0$, q.e.d.

4. Transversal Jacobi fields. Let now g be a transversal geodesic. We put $g(0)=a$; then g is of the form $a \cdot e^{tX}$, where e^{tX} is the 1-parameter group with initial vector X. By (3.6), the vector X belongs to $\mathfrak{p}_a$.

LEMMA 4.1. *Any transversal Jacobi field η along g with $\eta(0)=0$ can be written as*

$$\eta(t) = a \cdot e^{tX} \cdot (\mathrm{Ad}e^{-tX} \cdot Y - Y + tZ) \quad \textit{with} \quad Y \in \mathfrak{k}_a, Z \in \mathfrak{p}_a \cap \mathfrak{g}_X.$$

Proof. There are two obvious ways of varying g, with $g(0)$ fixed: conjugation of e^{tX} by elements y of G and multiplication of e^{tX} by e^{tZ} with any Z that commutes with X. To keep the geodesics transversal, we restrict y to $K \cap a^{-1} \cdot K \cdot a$ and Z to $\mathfrak{p}_a$. We shall show that with these two processes, one can produce all transversal variations with $g(0)$ fixed, and the associated J-fields are of the form given in 4.1.

Let then Y be any element of $\mathfrak{k}_a$, and Z any element of $\mathfrak{p}_a \cap \mathfrak{g}_X$. We consider the geodesic $g_{Y,Z}$, given by $a \cdot e^Y \cdot e^{t(X+Z)} \cdot e^{-Y}$, with $g_{Y,Z}(0)=a$, and with initial tangent vector $a \cdot \mathrm{Ad}e^Y \cdot (X+Z)$. Since $X \in \mathfrak{p}_a$, and $e^Y \in K \cap a \cdot K \cdot a^{-1}$ (because of $Y \in \mathfrak{k} \cap \mathrm{Ad}a^{-1} \cdot \mathfrak{k}$) this vector belongs to $a \cdot \mathfrak{p}_a$, and $g_{Y,Z}$ is transversal. Assigning $a \cdot \mathrm{Ad}e^Y \cdot (X+Z)$ to the pair (Y,Z) constitutes a map ϕ of $\mathfrak{k}_a \oplus (\mathfrak{p}_a \cap \mathfrak{g}_X)$ into $a \cdot \mathfrak{p}_a$, with $\phi(0,0) = a \cdot X$. Using (1.2(d)), one derives that the differential ϕ_0 at $(0,0)$ maps the pair (Y,Z) into $a \cdot [Y,X] + a \cdot Z$ (we have made use of the customary identification for tangent spaces to vector spaces).

Lemma 3.8 implies that ϕ_0 is a map onto $a \cdot \mathfrak{p}_a$. From the remarks on

J-fields in Chapter I, no. 6, taking $a \cdot \mathfrak{p}_a$ as the space W there, one concludes now that the map which to the vector (Y, Z) assigns the J-field of the variation $a \cdot e^{aY} \cdot e^{t(X+aZ)} e^{-aY}$ is a map of $\mathfrak{k}_a \oplus (\mathfrak{p}_a \cap \mathfrak{g}_X)$ onto the space of transversal J-fields vanishing for $t = 0$. One verifies by straight forward differentiating (using $[X, Z] = 0$) that this J-field is of the type described in (4.1), and so (4.1) is proved.

5. The infinitesimal motions. We first prove

LEMMA 5.1. *The infinitesimal $K \times K$-motions, which vanish at a, restrict along g to the vector fields of the form $a \cdot e^{tX} \cdot (\mathrm{Ad}e^{-tX} \cdot V - V)$ with $V \in \mathfrak{k}_a$.*

Proof. All 1-parameter $K \times K$-motions are clearly of the form $y \to e^{aU} \cdot y \cdot e^{-aV}$, with $U, V \in \mathfrak{k}$. The corresponding infinitesimal motion assigns to y the vector $U \cdot y - y \cdot V$. Such a field will vanish at a, if $U \cdot a - a \cdot V = 0$, i.e., if $U = \mathrm{Ad}a \cdot V$. In other words, the infinitesimal motions that vanish at a are of the form $(\mathrm{Ad}a \cdot V)y - y \cdot V$, with $V \in \mathfrak{k}$ and $\mathrm{Ad}a \cdot V \in \mathfrak{k}$, i.e with $V \in \mathfrak{k}_a$. For $y = a \cdot e^{tX}$, along g, this becomes $a \cdot e^{tX} \cdot (\mathrm{Ad}e^{-tX} \cdot V - V)$, q.e.d.

We come now to the proof of Proposition 2.2. Let η be as described there, and let Y, Z be the vectors associated with η by (4.1). The assumption that $\eta(1)$ is tangent to the orbit of $b = a \cdot e^X$ at b means, by (3.6),

$$b^{-1} \cdot \eta(1) = \mathrm{Ad}e^{-X} \cdot Y - Y + Z \in \mathfrak{k}^b. \tag{5.2}$$

Any vector B in $\mathfrak{k}^b$ can be written as $B_1 + B_2$ with $B_1 \in \mathfrak{k}$ and $B_2 \in \mathrm{Ad}b^{-1} \cdot \mathfrak{k}$; we can write B_2 as $\mathrm{Ad}e^{-X} \cdot B_2'$ with $B_2' \in \mathrm{Ad}a^{-1} \cdot \mathfrak{k}$, since $b = a \cdot e^X$. By (3.5), Z is orthogonal to $\mathfrak{k}^a$, so that the inner products (Z, B_1) and (Z, B_2') vanish. We have $\mathrm{Ad}e^X \cdot Z = Z$, since $[X, Z] = 0$. It follows that

$$(Z, B_2) = (\mathrm{Ad}e^X \cdot Z, \mathrm{Ad}e^X \cdot B_2) = (Z, B_2') = 0.$$

This means that $(Z, B) = 0$ and Z is orthogonal to $\mathfrak{k}^b$. From (5.2), we have then $0 = (\mathrm{Ad}e^{-X} \cdot Y - Y + Z, Z) = (\mathrm{Ad}e^{-X} \cdot Y, Z) - (Y, Z) + (Z, Z)$. Since $(\mathrm{Ad}e^{-X} \cdot Y, Z) = (Y, Z)$, we conclude $(Z, Z) = 0$, and so $Z = 0$. But then η is given by $a \cdot e^{tX} \cdot (\mathrm{Ad}e^{-tX} \cdot Y - Y)$, and this is, by (5.1), an infinitesimal motion, since $Y \in \mathfrak{k}_a$.

6. K operating on G/K. We consider now Theorem 11 for the action of K on G/K. We will reduce this case, with the help of the projection $p: G \to G/K$, to the previous case of $K \times K$ acting on G. It is well known and easily proved that for the Riemannian metric in G/K, as defined in no. 1,

the geodesics in G/K are exactly the projections of those geodesics in G that are orthogonal to the left cosets of K (i.e. to the geodesics which are transversal with respect to the action of K on G, defined by multiplying on the right). For any $a \in G$, the projection of the $K \times K$-orbit $K \cdot a \cdot K$ is the K-orbit $K \cdot a'$, with $a' = p(a)$. The space $a \cdot \mathfrak{p}_a$, the orthogonal complement of the tangent space to $K \cdot a \cdot K$ at a in the tangent space to G at a, is mapped by p isometrically onto the orthogonal complement of the tangent space at a' of the orbit $K \cdot a'$. It follows that the $K \times K$-transversal geodesics at a map onto the K-transversal geodesics at a' (in a 1-1 way), and that for any such geodesic at a, the space of transversal Jacobi fields that vanish at a maps onto the space of transversal J-fields along the projected geodesic that vanish at a'; moreover, any such J-field is tangent to the $K \times K$-orbit at a point b. if and only if the projected J-field is tangent to the K-orbit at $b' = p(b)$. In other words, p establishes an isomorphism between $\Lambda_g(0) \cap J_g{}^{\pi}(1)$ and $\Lambda_{g'}(0) \cap J_{g'}{}^{\pi}(1)$ for any transversal geodesic g in G and its projection g' in G/K. Finally, the projection is equivariant with respect to the natural map of $K \times K$ onto its first factor; i.e. $p((k_1, k_2) \cdot a) = k_1 \cdot p(a)$. It follows easily that any infinitesimal $K \times K$-motion along g that vanishes at a projects into an infinitesimal K-motion along g' that vanishes at a'. Since all elements in $\Lambda_g(0) \cap J_g{}^{\pi}(1)$ are induced by $K \times K$, it follows now that all elements in $\Lambda_{g'}(0) \cap J_{g'}{}^{\pi}(1)$ are induced by K, q.e.d.

7. The infinitesimal case. We come to the adjoint action of K on $\mathfrak{p}$, the infinitesimal analogue of the case just treated. We give a proof analogous to that of Proposition 2.2; another possible approach consists in comparing $\mathfrak{p}$ with a small K-invariant neighborhood of e' in G/K. Let A be any point in $\mathfrak{p}$.

PROPOSITION 7.1. *The tangent space to the orbit of A under the adjoint action of K is* $\mathrm{ad}A(\mathfrak{k})$.

This follows from the relation (1.2(d)).

DEFINITION 7.2. $\mathfrak{k}_A = \mathfrak{k} \cap \mathfrak{g}_A$; $\mathfrak{p}_A = \mathfrak{p} \cap \mathfrak{g}_A$ (*here $\mathfrak{g}_A$ is the centralizer of A*).

We have the relations:

$$[\mathfrak{k}_A, \mathfrak{p}_A] \subset \mathfrak{p}_A; \quad [\mathfrak{p}_A, \mathfrak{p}_A] \subset \mathfrak{k}_A; \tag{7.3}$$

$$\mathfrak{p}_A = \mathrm{ad}A(\mathfrak{k})^{\perp} \text{ (orthogonal complement in } \mathfrak{p}). \tag{7.4}$$

For the proof of (7.3), we use (3.1) and note that by the Jacobi identity,

the left sides are contained in $\mathfrak{g}_A$. To prove (7.3), take $P \in \mathfrak{p}$ and $Q \in \mathfrak{k}$. The relation $(P,[A,Q]) = -([A,P],Q)$ together with $[A,P] \in \mathfrak{k}$ (by (3.1)) implies the stated result. Note that $A \in \mathfrak{p}_A$.

Recall that the geodesics in $\mathfrak{p}$ are simply the straight lines. Let X by any element of $\mathfrak{p}_A$ and define the geodesic g by $g(t) = A + tX$; with X varying through $\mathfrak{p}_A$, this represents all the K-transversal geodesics starting at A.

LEMMA 7.5. $\mathfrak{p}_A = \mathrm{ad}X(\mathfrak{k}_A) \oplus (\mathfrak{p}_A \cap \mathfrak{g}_X)$ (*orthogonal decomposition*). The proof is analogous to that of (3.8) with (7.3), (7.4) replacing (3.4), (3.5).

LEMMA 7.6. *Any transversal J-field η along g with $\eta(0) = 0$ can be written as $\eta(t) = t(E_1 + E_2)$ with $E_1 \in \mathrm{ad}X(\mathfrak{k}_A)$, $E_2 \in \mathfrak{p}_A \cap \mathfrak{g}_X$.*

Proof. The transversal geodesics are the straight lines of the form $A + tC$, with $C \in \mathfrak{p}_A$, by 7.4. For the study of transversal Jacobi fields vanishing at A, it is clearly sufficient to consider linear variations, i.e. families $\{V_\alpha\}$ with $V_\alpha(t) = A + t \cdot (X + \alpha C)$ for $C \in \mathfrak{p}_A$. The Jacobi field of such a variation is given by tC. Splitting C according to Lemma 7.5, we get Lemma 7.6.

LEMMA 7.7. *The infinitesimal K-motions along g, vanishing at A, are exactly the fields of the form $t[Y,X]$ with $Y \in \mathfrak{k}_A$.*

Proof. The infinitesimal K-motion in $\mathfrak{p}$, determined by the 1-parameter group $\mathrm{Ad}e^{\alpha Y}$, assigns to the point S of $\mathfrak{p}$ the vector $[Y,S]$. Such a field vanishes at A exactly if $Y \in \mathfrak{k}_A$, and the field along g, i.e. for $S = A + tX$, is then $t[Y,X]$.

We prove now that variational completeness holds.

PROPOSITION 7.8. *Let η be a transversal Jacobi field along g with $\eta(0) = 0$ that is tangent to the K-orbit at $B = A + X$ (i.e., for $t = 1$); then η is induced by an infinitesimal K-motion.*

Decompose $\eta(t)$ into $t[X,Y] + tZ$ with $Y \in \mathfrak{k}_A$, $Z \in \mathfrak{p}_A \cap \mathfrak{g}_X$ by Lemma 7.6. To say that $\eta(1)$ is tangent to the orbit is to say that $[X,Y] + Z \in \mathrm{ad}(A+X)(\mathfrak{k})$, by (7.1). We have $([X,Y],Z) = -(Y,[X,Z]) = 0$ because of $[X,Z] = 0$. Take any $D \in \mathfrak{k}$ and form $([A+X,D],Z)$. This equals $-(D,[A,Z]) - (D,[X,Z])$. Since $[A,Z] = [X,Z] = 0$, we conclude that Z is orthogonal to $\mathrm{ad}(A+X)(\mathfrak{k})$. It follows now that $(Z,Z) = 0$, and so $Z = 0$. The proof of Theorem II is complete.

8. G as symmetric space. A well known case of symmetric pairs is obtained by starting with any compact connected Lie group G, forming $G \times G$, and defining an involution in $G \times G$ by $(x,y)^* = (y,x)$. The fixed group is then the diagonal Δ, isomorphic to G under $x \to (x,x)$. The quotient space $(G \times G)/\Delta$ can be identified with G through the map $(x,y) \to x \cdot y^{-1}$. The action of Δ on $(G \times G)/\Delta$ goes over into the action of G on itself by inner automorphisms. Similarly, the adjoint action of Δ on the tangent space of $(G \times G)/\Delta$ at the image of (e,e) (or on the subspace $\{(X,-X) : X \in \mathfrak{g}\}$ of $\mathfrak{g} \oplus \mathfrak{g}$) goes over into the adjoint action of G on $\mathfrak{g}$. Theorem II shows that both these actions are variationally complete; these two cases constitute theorems proved in [6], of which our Theorem II is then a generalization.

Chapter III. Applications to Certain Homogeneous Spaces and Loop Spaces.

1. The diagram of K, review of notation. Before stating, in no. 2, our main results, we collect here some facts about Lie groups (cf. [6], [20]).

K is again a compact connected Lie group, with Lie algebra $\mathfrak{k}$; in this chapter, we assume K semi-simple and simply connected. T is a maximal torus in K, with Lie algebra $\mathfrak{t}$; $\dim T = \operatorname{rank} K = l$. On $\mathfrak{k}$ we have the Killing form $(\ ,\ )$, invariant under the adjoint maps $\mathrm{Ad}x$, for all x in K. The orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}$ is denoted by $\mathfrak{m}$; it splits into m planes of dimension 2, invariant and irreducible under $\mathrm{Ad}T$. If $\mathfrak{e}$ is such a plane with a definite orientation, then there exists a unique linear function $\theta : \mathfrak{t} \to \boldsymbol{R}$, the root belonging to $\mathfrak{e}$, such that $\mathrm{Ad}e^X$, for any $X \in \mathfrak{t}$, operates on $\mathfrak{e}$ as rotation through the angle $2\pi\theta(X)$. It is possible to make a simultaneous choice of orientations for the m planes in such a way that among the corresponding roots θ there are l "fundamental" roots with the property that every θ is a non-negative integral linear combination of the fundamental ones. We assume such a choice made, and call $\mathfrak{R} = \{\theta\}$ the corresponding set of m roots. The dominant root, with all coefficients maximal, we denote by μ. We introduce the set $\mathfrak{P} = \{p\}$ of oriented singular planes in $\mathfrak{t}$; such a plane is by definition a pair (θ, n) of a root θ in $\mathfrak{R}$ and an integer n; for any such pair p, we write p also for the set $\{X \in \mathfrak{t} : \theta(X) = n\}$ and $\bar{p}$ for the image $\exp(p)$ under the exponential map into T or K. Each θ in $\mathfrak{R}$ defines a basic translation τ_θ, the element of $\mathfrak{t}$ that is orthogonal to the plane $(\theta, 0)$ and satisfies $\theta(\tau_\theta) = 2$. We write $\mathfrak{T}$ for the lattice in $\mathfrak{t}$ that the τ_θ generate; $\mathfrak{T}$ is identical with the kernel of the exponential map of $\mathfrak{t}$ into T. For any $p = (\theta, n) \in \mathfrak{P}$, we write $\theta_p = \theta$, $n_p = n$ and $\tau_p = \tau_\theta$. The Cartan integers are defined as the values

(1.1) $\theta_p(\tau_q)$, for $p, q \in \mathfrak{P}$;

only $0, \pm 1, \pm 2, \pm 3$ occur.

The diagram $D(K)$ is the union of all p in $\mathfrak{P}$; the infinitesimal diagram $D'(K)$ is the union of all $p \in \mathfrak{P}$ with $n_p = 0$. The components of $\mathfrak{t} - D(K)$ are the cells, Δ; the components of $\mathfrak{t} - D'(K)$ are the chambers. We write $\mathfrak{F}$ for the fundamental chamber in which all $\theta \in \mathfrak{R}$ take positive values, and $\Delta_{\mathfrak{F}}$ for the cell in $\mathfrak{F}$ that has the origin 0 in its closure.

Let N_T be the normalizer, in K, of T: The Weyl group $\mathfrak{W}(K)$ or $\mathfrak{W}$ is the (finite) factor group N_T/T. It operates on $\mathfrak{t}$, via the adjoint map, as a group of orthogonal transformations, generated by the reflections across the planes of $D'(K)$; it also operates on T and on K/T (cf. [1]).

2. The main results. The proofs of the results stated here appear in nos. 3-15.

Let s be a line segment in $\mathfrak{t}$, from a point R interior to the fundamental chamber $\mathfrak{F}$ to a point R' interior to the chamber $-\mathfrak{F}$, such that no point of s lies on two or more of the planes of $D'(K)$. Let the roots in $\mathfrak{R}$ be numbered θ_i, $1 \leqq i \leqq m$, in such a way that s meets the planes $(\theta_i, 0)$ in order of increasing index from R to R'. Write τ_i for τ_{θ_i}. Define a graded algebra A_K by

(2.1) $$A_K = \mathbf{Z}[x_1, \cdots, x_m]/I_m,$$

where $\mathbf{Z}[x_1, \cdots, x_m]$ is the polynomial ring over $\mathbf{Z}$ in the variables $x_1, \cdots, x_m$, each x_i of dimension 2, and where I_m is the ideal generated by the elements

(2.2) $$\rho_k = x_k \cdot (x_k + \sum_1^{k-1} \theta_k(\tau_i) x_i), \qquad 1 \leqq k \leqq m;$$

we denote the image of x_i in A_K again by x_i.

Denote by $\mathfrak{J}$ the module of integral forms on $\mathfrak{t}$, i.e. the set of those linear functions on $\mathfrak{t}$ that take integal values at all points of the lattice $\mathfrak{T}$. Finally, let B be the submodule of the linear (two-dimensional) part of A_K formed by the elements of the form $\sum \phi(\tau_i) x_i$ with ϕ ranging over $\mathfrak{J}$.

THEOREM III. *The integral cohomology ring $H^*(K/T)$ of the coset space K/T can be identified with the smallest subring of A_K that* (1) *contains the unit and the module B, and* (2) *is additively a direct summand. In particular, $H^2(K/T)$ is identified with B.*

It is easily seen that B can also be described as follows: Let $\{\zeta_1, \cdots, \zeta_l\}$ be a basis of the lattice $\mathfrak{T}$, and write $\tau_i = \sum a_{ij}\zeta_j$, $1 \leqq i \leqq m$. Then B has as a basis the elements $z_j = \sum a_{ij} x_i$, $1 \leqq j \leqq l$.

We note that this theorem describes the integral cohomology of K/T in terms of infinitesimal invariants of K, namely the Cartan integers and what amounts to the integral relations between the basic translations τ_i (for $\phi \in \mathfrak{J}$, the values $\phi(\tau_i)$ satisfy $\sum a_i \phi(\tau_i) = 0$ whenever $\sum a_i \tau_i = 0$ with real, necessarily rational, a_i).

As an example, we compute G_2/T for the exceptional group G_2. The results III and III′ were announced in [7].

THEOREM III′. *The cohomology ring $H^*(G_2/T)$ contains two elements α, β of dimension 2 such that $H^*(G_2/T)$ has an (additive) basis consisting of 1, α, β, $\alpha\cdot\beta$, β^2, $\alpha\cdot\beta^2$, $\beta^3/2$, $\alpha\cdot\beta^3/2$, $\beta^4/2$, $\alpha\cdot\beta^4/2$, $\beta^5/2$, $\alpha\cdot\beta^5/2$; moreover, the relations $\alpha^2 + 3\alpha\cdot\beta + 3\beta^2 = 0$, $\alpha^6 = \beta^6 = 0$ hold.*

Here $\beta^3/2$ means of course an element x such that $2x = \beta^3$.

For any group K, one has the natural action of the Weyl group $\mathfrak{W}$ on K/T and on $H^*(K/T)$. In the description of $H^*(K/T)$, given by Theorem III, this action takes the following form: The operations of $\mathfrak{W}$ on $\mathfrak{t}$ map the lattice $\mathfrak{T}$ into itself. Consequently, there is an induced operation on the module $\mathfrak{J}$ of integral linear forms introduced before Theorem III. Finally, $\mathfrak{W}$ acts on the submodule B of A_K, defined there, since B is naturally isomorphic to $\mathfrak{J}$.

PROPOSITION 2.1. *Under the identification of B and $H^2(K/T)$ of Theorem III, the action of $\mathfrak{W}$ on B becomes identical with the action of $\mathfrak{W}$ on $H^2(K/T)$ derived from the action of $\mathfrak{W}$ on K/T. The action on B extends uniquely to the action on $H^*(K/T)$, the latter considered as a subalgebra of A_K.*

In terms of the bases $\{\zeta_i\}$, $\{z_i\}$ above, we express the image $w\cdot\zeta_i$ of ζ_i under the action of $w \in \mathfrak{W}$ as $\sum b_{ij}\zeta_j$; then the relation $w\cdot z_j = \sum b_{ij} z_i$ defines the operation of w on $H^2(K/T)$.

Let now T' be any non-maximal torus of K, and let $K_{T'}$ (denoted by $C(T')$ in [6]) be the centralizer of T' (a connected group by ([10]). We may assume $T' \subset T$; the latter is then a maximal torus of $K_{T'}$, and the Weyl group $\mathfrak{W}' = \mathfrak{W}(K_{T'})$ is a subgroup of $\mathfrak{W}(K)$.

THEOREM III″. *The cohomology ring $H^*(K/K_{T'})$ is isomorphic to the subalgebra of $H^*(K/T)$, consisting of the invariants of $\mathfrak{W}'$ in $H^*(K/T)$.* (The action of $\mathfrak{W}$, and therefore also of $\mathfrak{W}'$, on $H^*(K/T)$ has been described in Proposition 2.1.).

We note that Theorem III, Proposition 2.1 and Theorem III″ together

yield a description of $H^*(K/K_{T'})$ in terms of the Cartan integers, the relations between the basic translations, and the action of the Weyl group of $K_{T'}$ on the basic translations.

The following theorems concern the homotopy groups π_i of Lie groups. It is a classical fact that $\pi_2(K) = 0$ for semi-simple K; and it is known [6] that $\pi_3(K) = \mathbf{Z}$ for simple K. We use the standard notation A_n, B_n, C_n, D_n for the classical groups, and G_2, F_4, E_6, E_7, E_8 for the exceptional groups.

THEOREM IV. *Let K be simple and let μ be its dominant root.*

(a) *$\pi_4(K)$ is either 0 or $\mathbf{Z}_2$; it is 0 exactly if the plane $(\mu, 1)$ with equation $\mu(X) = 1$ in $\mathfrak{t}$ contains a point of the basic lattice $\mathfrak{I}$.*

(b) *The groups A_1, B_1, and C_n, $n \geqq 1$, have $\pi_4 = \mathbf{Z}_2$, all other simple groups have $\pi_4 = 0$.*

THEOREM V.

(a) $\pi_i(E_6) = 0$ for $4 \leqq i \leqq 8$; $\pi_9(E_6) = \mathbf{Z}$;

(b) $\pi_i(E_7) = 0$ for $4 \leqq i \leqq 10$; $\pi_{11}(E_7) = \mathbf{Z}$;

(c) $\pi_i(E_8) = 0$ for $4 \leqq i \leqq 14$; $\pi_{15}(E_8) = \mathbf{Z}$;

(d) $\pi_{10}(G_2) \otimes \mathbf{Z}_3 = 0$.

Part (d) is a correction of the value given in Toda's table in [4]; it is in agreement with recent calculations of Toda, and also with recent results of A. Borel-F. Hirzebruch (not yet published).

3. The groups K_p. The proofs of the theorems of no. 2 depend on the construction of certain iterated 2-sphere bundles that appear as K-cycles for suitable actions of K.

For each root $\theta \in \mathcal{R}$, let K_θ be the stabilizer of the singular plane $(\theta, 0)$ of $\mathfrak{t}$, under the adjoint action of K on $\mathfrak{k}$; denote by $\mathcal{K}$ the set of K_θ's. For any oriented singular plane $p \in \mathcal{P}$, we write K_p for the centralizer of $\bar{p}$ (stabilizer under the action of K on itself by inner automorphisms). It is familiar that for $p = (\theta, n)$, we have $K_p = K_\theta$, independent of n, and that K_p can be described as follows:

Let S_3 denote the special unitary group $SU(2)$ (or equivalently, the group of quaternions of norm 1), topologically a 3-sphere; let S_1 denote the circle group contained in S_3, consisting of the ordinary complex numbers $\cos\phi + i\sin\phi$ in the quaternion form of S_3. For each $p \in \mathcal{P}$, there exists a monomorphism

$$\beta_p : S_3 \to K_p \tag{3.1}$$

depending only on θ_p and not on n_p, such that T and $\beta_p(S_3)$ generate K_p and such that $\beta_p(S_3) \cap T = \beta_p(S_1)$, cf. [11]. (That β_p is a monomorphism depends on the fact that the element $\frac{1}{2} \cdot \tau_p$ does not belong to the lattice $\mathcal{J}$, and that therefore $\beta_p(-1) \neq e$.) S_3 is a principal bundle for its subgroup S_1, with base space S_2, the 2-sphere (Hopf fibering). Similarly, K_p is a principal T-bundle; β_p is a bundle map, and induces a homeomorphism β_p of the base spaces; we note explicitly that every K_p/T is a 2-sphere.

Let j denote the quaternion unit usually denoted by that letter, and put $j_p = \beta_p(j) \in K_p$. The inner automorphism of K by j_p maps T into itself; its restriction to T is an involution which we denote by R_p. Similarly, $\mathrm{Ad} j_p$ maps $\mathfrak{t}$ into itself; its restriction to $\mathfrak{t}$ is also an involution, again denoted by R_p, and given by the formula

$$R_p(X) = X - \theta_p(X)\tau_p, \text{ for } X \in \mathfrak{t} \tag{3.2}$$

(reflection across the plane $(\theta_p, 0)$). The R_p's generate the Weyl group.

Using the standard isomorphism of $\pi_1(T)$ and $H_1(T)$, and the identification of $\pi_1(T)$ with the group $\mathcal{J}$ of covering translations of the universal covering space $\mathfrak{t}$ of T, we associate to each basic translation τ_p or τ_θ a homology class $\boldsymbol{\tau}_p$ or $\boldsymbol{\tau}_\theta$ in $H_1(T)$. For the homology map, induced by R_p, we have then from (3.2),

$$R_{p*}(\boldsymbol{\tau}_q) = \boldsymbol{\tau}_q - \theta_p(\tau_q)\boldsymbol{\tau}_p. \tag{3.3}$$

In particular, on S_1 we have a standard generator $\boldsymbol{\tau}$ of $H_1(S_1)$, such that for the $\beta_p\colon S_1 \to T$, associated with any $p \in P$,

$$\beta_{p*}(\boldsymbol{\tau}) = \boldsymbol{\tau}_p \in H_1(T). \tag{3.4}$$

4. The manifolds Γ_P. Let $P = \{p_1, \cdots, p_r\}$ be an ordered finite sequence of oriented singular planes in $\mathfrak{P}$. Using definition 4.2 of Chapter I, we put

$$\Gamma_P = K_{p_1} \times_T K_{p_2} \times_T \cdots \times_T K_{p_{r-1}} \times_T K_{p_r}/T. \tag{4.1}$$

Our aim is to compute the cohomology of Γ_P. We abbreviate K_{p_i} to K_i, and similarly, θ_{p_i} to θ_i, etc. The polynomial ring $\mathbf{Z}[x_1, \cdots, x_r]$ will be considered as a graded ring, the variables having dimension 2.

PROPOSITION 4.2. *The cohomology ring $H^*(\Gamma_P)$ is isomorphic to $\mathbf{Z}[x_1, \cdots, x_r]/I_P$, where I_P is the ideal generated by the elements*

$$\rho_k = x_k \cdot (x_k + \sum_1^{k-1} \theta_k(\tau_i)x_i), \qquad 1 \leqq k \leqq r.$$

In particular, $H^*(\Gamma_P)$ has no torsion and is generated by 2-dimensional elements.

The proof will occupy nos. 4 and 5. The proposition is obvious for $r=1$, since then Γ_P is a 2-sphere (cf. no. 3). We proceed by an inductive construction, and take $r>1$. As in Chapter I, (4.6), we put

$$W_P = K_1 \times \cdots \times K_r. \tag{4.3}$$

The space Γ_P is then also the base space of the principal action of $T^r = T \times \cdots \times T$ (r factors) on W_P, given by

$$(k_1, \cdots, k_r) \cdot (t_1, \cdots, t_r) = (k_1 t_1, t_1^{-1} k_2 t_2, \cdots, t_{r-1}^{-1} k_r t_r),$$

where $k_i \in K_i$, and $t_i \in T$; let ψ be the projection of W_P onto Γ_P.

Put $P' = \{p_1, \cdots, p_{r-1}\}$, defining $\Gamma_{P'}$, $W_{P'}$ and ψ'. Let $\tilde{\phi}: W_P \to W_{P'}$ be defined by suppressing the last coordinate. This is a bundle map, relative to the homomorphism $\phi: T^r \to T^{r-1}$, defined similarly, and induces a map $\bar{\phi}: \Gamma_P \to \Gamma_{P'}$. Writing T^r as $T^{r-1} \times T$, we can factor the projection ψ by first operating on W_P with the factor T, which results in $W_{P'} \times K_r/T$, and then operating with T^{r-1}. One reads off from the basic diagram of Chapter I, (3.1) that Γ_P is the K_r/T-bundle over $\Gamma_{P'}$, associated to the action of T^{r-1} on K/T, given by $(t_1, \cdots, t_{r-1}) \cdot k_r T = t_{r-1} k_r T$. Let $\tilde{s}$ be the natural injection of $W_{P'}$ into $W_P = W_{P'} \times K_r$. One verifies that this is a bundle map, relative to the natural injection of T^{r-1} into $T^r = T^{r-1} \times T$; denote by $\bar{s}$ the induced map of $\Gamma_{P'}$ into Γ_P. Since $\tilde{\phi} \circ \tilde{s} = \text{identity}$, the map $\bar{s}$ is a cross section of the bundle map $\bar{\phi}$.

We now set up suitable bases for homology and cohomology. Let $\tilde{\chi}_i$, for $1 \leqq i \leqq r$, denote the map of S_3 into W_P, defined by the β_i (cf. (3.1)) of S_3 into the factor K_i of W_P. This is a bundle map relative to the homomorphism $\chi_i: S_1 \to T$, given by

$$\chi_i(t) = (e, \cdots, e, \beta_i(t), \cdots, \beta_i(t)), \text{ with } i-1 \text{ } e\text{'s}. \tag{4.4}$$

Let $\bar{\chi}_i$ be the induced map of S_2 into Γ_P.

We shall apply the transgression map $\partial: H_2(\text{base}) \to H_1$ (fiber), for the various bundles under consideration; in all cases occurring here, ∂ can be interpreted, via the Hurewicz isomorphism, as the boundary map (dimension 2) of the homotopy sequence of the bundle. We define a generator y of $H_2(S_2)$ by $\partial y = \tau$, where τ is the standard generator of $H_1(T)$ introduced above (cf. (3.4)), and define elements y_i of $H_2(\Gamma_P)$ by

$$y_i = \bar{\chi}_{i*}(y), \qquad 1 \leqq i \leqq r. \tag{4.5}$$

One verifies immediately that the elements y_i, $1 \leqq i \leqq r-1$, map under $\bar{\phi}_*$ into the similarly defined elements y_i', $1 \leqq i \leqq r-1$, of $H_2(\Gamma_{P'})$, that y_r represents a generator of the second homology group of the fiber K_r/T of $\bar{\phi}$, so that $\bar{\phi}_*(y_r) = 0$, and that the y_i', $1 \leqq i \leqq r-1$, map into y_i, $1 \leqq i \leqq r-1$, under the cross section $\bar{s}$.

One concludes now inductively by a simple application of the Gysin sequence (for 2-sphere bundles with cross section) (cf. [18]) that the following proposition holds.

PROPOSITION 4.6. *$H^*(\Gamma_P)$ has no torsion (and Γ_P is simply connected). The elements y_i, $1 \leqq i \leqq r$, defined in (4.5), form a basis for $H_2(\Gamma_P)$. If ξ is any element of $H^2(\Gamma_P)$ whose restriction to the fiber K_r/T of $\bar{\phi}$ is a generator of $H^2(K_r/T)$, then the map $\bar{\phi}^*$, and the map consisting of $\bar{\phi}^*$ followed by cup-product with ξ are isomorphisms into, and define a direct sum decomposition*

$$H^*(\Gamma_P) = \bar{\phi}^* H^*(\Gamma_{P'}) \oplus \xi \cdot \bar{\phi}^* H^*(\Gamma_{P'}).$$

Specifically, the ring structure of $H^*(\Gamma_P)$ is determined by $H^*(\Gamma_{P'})$ and the expression for ξ^2 in this above decomposition.

Let $\{x_i\}_1^r$ be the basis of $H^2(\Gamma_P)$ dual to the basis $\{y_i\}$ of $H_2(\Gamma_P)$ introduced above, so that

$$x_i(y_j) = \delta_{ij}, \qquad 1 \leqq i, j \leqq r. \tag{4.7}$$

The element x_r can then serve as ξ in 4.6. From the properties of the y_i's, it follows at once that the elements x_i, $1 \leqq i \leqq r-1$, are the images under $\bar{\phi}^*$ of the similarly defined elements x_i', $1 \leqq i \leqq r-1$, of $\Gamma_{P'}$, and therefore generate $\bar{\phi}^* H^*(\Gamma_{P'})$, and that

$$\bar{s}^*(x_r) = 0. \tag{4.8}$$

5. The involution J. To find the expression for x_r^2, we make use of an involution on Γ_P, which reverses the orientation of the fiber. (This method goes back to [12], cf. also [14].) Let $\tilde{J}: W_P \to W_P$ be the map sending

$$(k_1, \cdots, k_r) \text{ into } (k_1, \cdots, k_{r-1}, k_r \cdot j_r^{-1}),$$

where $j_r = \beta_r(j)$ as defined in no. 3. This is a bundle map of the bundle ψ relative to the map $J: T^r \to T^r$, defined by $J(t_1, \cdots, t_r) = (t_1, \cdots, t_{r-1}, R_r(t_r))$, with R_r the involution of T determined by conjugation by j_r (cf. no. 3). Let $\bar{J}$ be the induced map of Γ_P into itself. Since $(j_r)^2$ belongs to T, $(J)^2$ is a bundle map over the identity and $(\bar{J})^2$ is the identity of Γ_P. Further, since $\tilde{J}$ operates only on the last coordinate of W_P, the map $\bar{J}$ is an

involution of the bundle Γ_P over the identity of the base $\Gamma_{P'}$:

$$\bar{\phi} \circ \bar{J} = \bar{\phi}. \tag{5.1}$$

To determine the effect of J on $H_2(\Gamma_P)$, we consider, for $1 \leqq i \leqq r$, the map $\tilde{\chi}_i : S_3 \to W_P$. We write again $T^r = T^{r-1} \times T$; using e_0 as generic symbol for the positive generator of the 0-th homology group (a point with coefficient 1), we have $H_1(T^r) = H_1(T^{r-1}) \otimes e_0 + e_0 \otimes H_1(T)$ [Künneth]. From (4.5), (4.4) and (3.4), we conclude that

$$\partial y_i = \tilde{\chi}_{i*}(\tau) = \tau'_i \otimes e_0 + e_0 \otimes \tau_i, \tag{5.2}$$

where the τ'_i are certain elements of $H_1(T^{r-1})$, with $\tau'_r = 0$. (We have used the fact that the composition of χ_i with the projection of $T^{r-1} \times T$ onto T is just β_i.)

Equivariance with respect to J and $\bar{J}$ shows that

$$\partial \circ \bar{J}_*(y_i) = \tau'_i \otimes e_0 + e_0 \otimes R_{r*}(\tau_i),$$

since J operates only on the last factor of T^r, as R_r. By (3.3), we have

$$\partial(\bar{J}_* y_i) = \tau'_i \otimes e_0 + e_0 \otimes \tau_i - \theta_r(\tau_i) e_0 \otimes \tau_r = \partial(y_i - \theta_r(\tau_i) y_r).$$

The well known fact "$\pi_2(W_P) = 0$" implies that ∂ is an isomorphism into, and we conclude that

$$\bar{J}_*(y_i) = y_i - \theta_r(\tau_i) y_r, \qquad 1 \leqq i \leqq r. \tag{5.3}$$

We note that for $i = r$, this means, because of $\theta_r(\tau_r) = 2$,

$$\bar{J}_*(y_r) = -y_r,$$

so that $\bar{J}$ reverses the orientation of the fiber K_r/T of Γ_P. Going to the dual basis $\{x_i\}$ for $H^2(\Gamma_P)$, we find

$$\bar{J}^*(x_r) = -x_r - \sum_1^{r-1} \theta_r(\tau_i) x_i. \tag{5.4}$$

It follows from the remark at the end of no. 4 that $\bar{J}^*(x_r)$ is congruent to $-x_r \bmod \bar{\phi}^* H^*(\Gamma_{P'})$. Using x_r as the ξ in (4.6), one concludes easily (using also (5.1)) that the elements of $H^*(\Gamma_P)$, invariant under $\bar{J}^*$, are exactly the elements in $\bar{\phi}^* H^*(\Gamma_P)$. We consider now $v = x_r \cdot \bar{J}^*(x_r)$. Since $\bar{J}$ is an involution and $\dim x_r = 2$, we have $\bar{J}^*(v) = v$, and by what was just said, v is of the form $\bar{\phi}^*(u)$. By (4.8), we have $\bar{s}^*(v) = 0$, and therefore $u = \bar{s}^* \circ \bar{\phi}^*(u) = \bar{s}^*(v) = 0$ (since $\bar{\phi} \circ \bar{s} = \text{identity}$), and finally $v = 0$. Writing out $x_r \cdot \bar{J}(x_r)$ from (5.4), we obtain

$$x_r \cdot (x_r + \sum_1^{r-1} \theta_r(\tau_i) x_i) = 0 \quad \text{or}$$

(5.5)

$$x_r^2 = -\sum_1^{r-1} \theta_r(\tau_i) x_i x_r.$$

An obvious induction argument, based on (4.6) and the description of $\bar{\phi}^* H^*(\Gamma_P)$, contained in the last sentence of no. 4, completes the proof of (4.2).

6. Description of K/T. We turn to the proof of Theorem III. For this purpose, we consider, as in [6], the adjoint action of K on $\mathfrak{k}$, and apply the considerations of Chapter I. The defect function δ of this action (Chapter I, Definition 7.1)), for points of $\mathfrak{t}$, is determined by the infinitesimal diagram $D'(K)$; namely, $\delta(Q)$ is equal to twice the number of planes in $D'(K)$ that contain Q. This follows readily from the definition of $D'(K)$ and Chapter I, (7.2). In particular, any point of $\mathfrak{t} - D'(K)$ lies on a maximal K-orbit. As the centralizer of such a point is precisely T, its orbit N can be identified with K/T. Let $\alpha: K/T \to N$ be such an identification, and consider the space $\Omega_\mathfrak{t} = \Omega(\mathfrak{k}; R, N)$, where R is a definite point of $\mathfrak{t} - D'(K)$ not on N. Let $\kappa: \Omega_\mathfrak{t} \to N$ be the natural projection of paths onto their end points. Because $\mathfrak{k}$ is a Euclidean space, κ induces a homotopy equivalence. Hence $\alpha^{-1} \circ \kappa$ defines a homotopy equivalence: $\Omega_\mathfrak{t} \approx K/T$. We will now apply Theorem I to $\Omega_\mathfrak{t}$. This is legitimate because of Theorem II.

We recall from [6] that N meets $\mathfrak{t}$ in a finite number of points $Q_1, \cdots, Q_w$ (one in each fundamental chamber). The set $S = S(\mathfrak{k}; R, N)$ consists of the straight line segments s_i from R to the Q_i, and δ_{s_i} is equal to twice the number of planes in $D'(K)$ crossed by s_i. We may assume that the s_i intersect the planes of $D'(K)$ one at a time.

The construction of Chapter I, no. 5, determines K-cycles Γ_i, and maps $f_i: \Gamma_i \to \Omega_\mathfrak{t}$, for each $s_i \in S$. The stabilizer of a generic point on the plane $(\theta_i, 0)$ is the group $K_{\theta_i} = K_i$ in the system $\mathfrak{K}$ defined in no. 3. The Γ_i are therefore of the type considered in no. 4; in particular, each Γ_i is orientable. Hence, by Theorem I, images of the fundamental classes of the Γ_i under $\alpha^{-1} \circ \kappa \circ f_i$ form a basis for the integral homology of K/T.

Since K/T is an orientable manifold it follows that the manifold Γ_{i_0} of highest dimension, $2m$, maps onto K/T with degree ± 1. This K-cycle, to be denoted by Γ_K, is characterized by the fact that R and Q_{i_0} lie in opposite fundamental chambers $\mathfrak{F}$ and $-\mathfrak{F}$. Adopting the numbering of the roots $\{\theta\}$, described at the beginning of no. 2, with $Q_{i_0} = R'$, we see from Definition 4.1 that

(6.1) $$\Gamma_K = K_1 \times_T K_2 \times_T \cdots \times_T K_{m-1} \times_T K_m/T,$$

The next proposition expresses a general property of maps of degree one; put $f_K = \alpha^{-1} \circ \kappa \circ f_{i_0} \colon \Gamma_K \to K/T$.

PROPOSITION 6.2. *The map f_K^* maps $H^*(K/T)$ isomorphically onto an additively-direct summand of $H^*(\Gamma_K)$.*

Proof. If γ' and γ are suitable fundamental cycles of Γ_K and K/T, we have $f_{K*}(\gamma') = \gamma$, expressing the fact that f_{i_0} is of degree one. We denote by d the Poincaré duality operator, i.e. the cap-product with the fundamental cycle. The permanence relation $f_{K*}(f_K^*(a) \cap \gamma') = a \cap f_{K*}(\gamma')$ implies then $f_{K*} \circ d \circ f_K^* = d$. Defining $\bar{f} \colon H^*(\Gamma_K) \to H^*(K/T)$ by $\bar{f} = d^{-1} \circ f_{K*} \circ d$, we get $\bar{f} \circ f_K^* = d^{-1} \circ d$ = identity, and the proposition follows by standard group theory.

Since $H^*(\Gamma_K)$ has no torsion, Proposition 6.2 gives another proof of the well known fact that K/T has no torsion. [See [6] for references.]

To give the promised description of $H^*(K/T)$, we now make use of an important proposition due to J. Leray and A. Borel [1].

PROPOSITION 6.3. *The cohomology ring $H^*(K/T; \boldsymbol{Q})$ over the rationals $\boldsymbol{Q}$ is generated by (the unit and) $H^2(K/T; \boldsymbol{Q})$.*

The two Propositions 6.2 and 6.3 and the well-known relations between rational and integral cohomology clearly imply the following proposition.

PROPOSITION 6.4. *$H^*(K/T)$ is isomorphic, under f_K^*, to the smallest subalgebra of $H^*(\Gamma_K)$ that contains the unit and $f_K^*(H^2(K/T))$ and is additively a direct summand.*

In Proposition 4.2, we have shown that $H^*(\Gamma_K)$ is the algebra A_K of Theorem III. The theorem will therefore follow from Proposition 6.4, once we have identified the f_K^*—image of $H^2(K/T)$ in $H^2(\Gamma_K)$. We begin by noting that the transgression ∂ in the bundle $K \to K/T$ maps $H_2(K/T)$ isomorphically onto $H_1(T)$, because of $\pi_1(K) = \pi_2(K) = 0$ (cf. no. 4). Instead of f_K^*, we consider $\partial \circ f_{K*}$.

PROPOSITION 6.5. *Let $\{y_i\}_1^m$ be the basis of $H_2(\Gamma_K)$, constructed in 4.5; the image of y_i in $H_1(T)$ under $\partial \circ f_{K*}$ is τ_i.*

Proof. Let ψ be the bundle projection of the T^m-bundle $W_K = K_1 \times \cdots \times K_m$ onto Γ_K, as in no. 4, and let g be the map of W_K into K, defined by $g(k_1, \cdots, k_m) = k_1 \cdot k_2 \cdot \cdots \cdot k_m$. One verifies that g is a bundle map relative to the projection of T^m onto its last factor T, with K as T-bundle in

the usual fashion, and that the induced map of the base spaces is just f_K. For $1 \leqq i \leqq m$, consider the map $\bar{\chi}_i \colon S_3 \to W_K$. The composition with g is a bundle map, relative to the map $\beta_i \colon S_1 \to T$, over the map $\beta_i \colon S_2 \to K/T$. It follows that

$$\partial \circ f_{K*}(y_i) = \partial \circ f_{K*} \circ \bar{\chi}_i(y) = \partial \circ \bar{\beta}_{i*}(y) = \beta_{i*}(\partial y) = \beta_{i*}(\tau) = \tau_i$$

(by (3.4)).

The map dual to $\partial \circ f_{K*}$ sends the cohomology class $\phi \in H^1(T)$ into the element $\sum \phi(\tau_i) x_i \in H^2(\Gamma_K)$. Each such ϕ can be identified with an integral linear form ϕ on $\mathfrak{t}$ such that $\phi(\tau_i) = \phi(\tau_i)$ (cf. the definition of $\mathfrak{J}$ before Theorem III and the remark before (3.3)). It is now clear that $f_K{}^* H^2(K/T)$ is the module B of Theorem III.

7. Computation of $H^*(G_2/T)$. As an example, we compute the cohomology of G_2/T, where G_2 is the exceptional group of rank 2. As is customary, we represent the Cartan algebra $\mathfrak{t}$ of G_2 as the plane with equation $t_1 + t_2 + t_3 = 0$ in Euclidean $t_1 t_2 t_3$-space. The 6-roots θ_i, ordered by the prescription of no. 2 and with a suitable choice of signs, are then given by the expression $t_1 - t_2, t_2, t_2 - t_3, -t_3, t_1 - t_3, t_1$. The corresponding vectors τ_i are given by $(1,-1,0)$, $(-1,2,-1)$, $(0,1,-1)$, $(1,1,-2)$, $(1,0,-1)$, $(2,-1,-1)$. The Cartan integers $\theta_r(\tau_s)$ are given by the matrix

$$(7.1) \qquad \begin{bmatrix} 2 & -3 & -1 & 0 & 1 & 3 \\ -1 & 2 & 1 & 1 & 0 & -1 \\ -1 & 3 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 & 2 & 3 \\ 1 & -1 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

The relations between the variables $x_1, \cdots, x_6$ in the algebra A_{G_2} are then the following:

$$(7.2) \qquad \begin{aligned} x_1{}^2 &= 0 \\ x_2{}^2 &= x_1 x_2 \\ x_3{}^2 &= x_1 x_3 - 3 x_2 x_3 \\ x_4{}^2 &= -x_2 x_4 - x_3 x_4 \\ x_5{}^2 &= -x_1 x_5 - x_3 x_5 - 3 x_4 x_5 \\ x_6{}^2 &= -x_1 x_6 + x_2 x_6 - x_4 x_6 - x_5 x_6. \end{aligned}$$

As basis for the lattice $\mathfrak{I}$ spanned by the τ_i, we can take τ_1 and τ_2. The matrix (a_{ij}) is then the matrix of the relations

$$
\begin{aligned}
\tau_1 &= \tau_1 \\
\tau_2 &= \tau_2 \\
\tau_3 &= \tau_1 + \tau_2 \\
\tau_4 &= 3\tau_1 + 2\tau_2 \\
\tau_5 &= 2\tau_1 + \tau_2 \\
\tau_6 &= 3\tau_1 + \tau_2.
\end{aligned} \tag{7.3}
$$

The generators for the image of $H^2(G_2/T)$ in A_{G_2} are therefore

$$
\begin{aligned}
a &= x_1 + x_3 + 3x_4 + 2x_5 + 3x_6 \\
b &= x_2 + x_3 + 2x_4 + x_5 + x_6.
\end{aligned}
$$

We change to other generators defined by

$$
\begin{aligned}
\alpha &= 3b - a = -x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 \\
\beta &= a - 2b = x_1 - 2x_2 - x_3 - x_4 + x_6.
\end{aligned} \tag{7.4}
$$

One verifies the relations

$$
\begin{aligned}
&\text{a)} \quad \alpha^2 + 3\alpha\beta + 3\beta^2 = 0, \\
&\text{b)} \quad \beta^3 \text{ divisible by } 2, \\
&\text{c)} \quad \alpha \cdot \beta^5 = 2x_1 \cdot x_2 \cdot \cdots \cdot x_6, \\
&\text{d)} \quad \alpha^6 = \beta^6 = 0.
\end{aligned} \tag{7, 5}
$$

Relation (7.5a)) shows that the products of the form β^t, $\alpha \cdot \beta^t$, $0 \leqq t \leqq 5$, span $f^*(H^*(G_2/T))$ rationally, i.e. up to division by integers. The product $x_1 \cdot \cdots \cdot x_6$ represents the fundamental cocycle of Γ_{G_2}. It follows from (7.5c)) that all divisibility relations in $H^*(G_2/T)$ are consequences of (7.5b)), and that a basis for $H^*(G_2/T)$ is given by

$$\{1, \alpha, \beta, \alpha \cdot \beta, \beta^2, \alpha \cdot \beta^2, \beta^3/2, \alpha \cdot \beta^3/2, \beta^4/2, \alpha \cdot \beta^4/2, \beta^5/2, \alpha \cdot \beta^5/2\}.$$

This completely determines the cohomology ring $H^*(G_2/T)$, and proves Theorem III′. The computations are facilitated by making the change of variables:

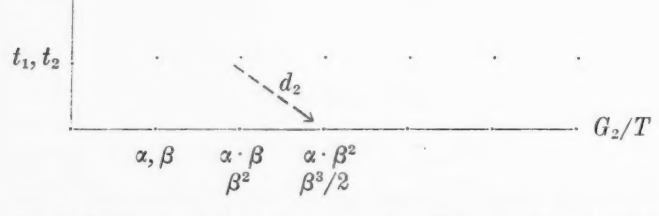
$$
\begin{aligned}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= -2x_1 + 3x_2 + x_3 \\
y_4 &= -x_1 + 2x_2 + x_3 + x_4 \\
\alpha &= -x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 \\
\beta &= x_1 - 2x_2 - x_3 - x_4 + x_6.
\end{aligned} \tag{7.6}
$$

The relations (7.2) go over into

$$
\begin{aligned}
y_1^2 &= 0 \\
y_2^2 &= y_1 y_2 \\
y_3^2 &= 3y_1y_2 - 3y_1y_3 + 3y_2y_3 \\
y_4^2 &= -\tfrac{1}{3}y_3^2 + y_3y_4 \\
\alpha^2 &= -3y_4^2 + 3y_4 \cdot \alpha \\
\beta^2 &= \tfrac{1}{3}\alpha^2 + \alpha\beta.
\end{aligned}
\tag{7.7}
$$

It is clear that $x_1 \cdots \cdots x_6 = y_1y_2y_3y_4\alpha\beta$.

As an application, one verifies, by a standard computation with the spectral sequence of $G_2 \to G_2/T$, that $H^*(G_2)$ has $\mathbf{Z}_2$ as torsion group in dimensions 6 and 9, as found by Borel in [3].



Note that the differential indicated above has $\mathbf{Z}_2$ as cokernel: $\beta^3/2$ is not in the image, although, with $d_2t_1 = \alpha$, $d_2t_2 = \beta$, one has $d_2(\beta^2 \cdot t_2) = 2 \cdot \beta^3/2$.

8. Cohomology of $K/K_{T'}$. We continue to use the notation introduced in no. 6. The proof of Proposition 2.1, to which we turn now, depends on the fact that the transgression $\partial: H_2(K/T) \to H_1(T)$, is equivariant with respect to the operation of the Weyl group, and that the same holds for the dual map ∂^* of the cohomology groups. This proves the statement about the action of $\mathfrak{W}$ on B, as a consequence of the remarks at the end of no. 6. The uniqueness of the extension to all of $H^*(K/T)$ follows from Theorem III. Note that $\mathfrak{W}$ does not act as a group of automorphisms on A_K.

The proof of Theorem III″ goes along quite different lines. We consider the spectral sequence of the fiber map $K/T \to K/K_{T'}$, with fiber $K_{T'}/T$, induced by the inclusions $K \supset K_{T'} \supset T$. All dimensions in $H^*(K/K_{T'})$ and in $H^*(K_{T'}/T)$ are even, and the spectral sequence is trivial (all differentials vanish). The Weyl group $\mathfrak{W}'$ of $K_{T'}$ operates on the spectral sequence. The operation is trivial on the base space $K/K_{T'}$, since $K_{T'}$ is connected. The operation on $H^*(K_{T'}/T)$ is known to be equivalent to the regular repre-

sentation of $\mathfrak{W}'$ (cf. [1]); the elements left fixed by all $w \in \mathfrak{W}'$ are exactly the elements of $H^0(K_{T'}/T)$. It follows now by comparing the E_2-term $H^*(K/K_{T'}) \otimes H^*(K_{T'}/T)$ with $H^*(K/T)$, that $H^*(K/K_{T'})$ [considered as a subalgebra of $H^*(K/T)$] consists exactly of the elements of $H^*(K/T)$ left fixed by all $w \in \mathfrak{W}'$, and Theorem III'' is proved.

9. The holomogy of $\Omega(K)$; applications to homotopy. Let K operate on itself by inner automorphisms. Let R be a point of K, and let $\Omega_R = \Omega(K; R, e)$ be the space of paths, as in Chapter I, no. 2, from R to the orbit of e, i.e. to e. Ω_R can be identified, in standard fashion, with the space $\Omega(K) = \Omega(K; e, e)$ of loops in K at e; since K is simply connected, this identification is unique up to homotopy.

Suppose now that R is a regular point of T, i.e. supposes $R \notin \bar{p}$ for any $p \in \mathfrak{P}$; the orbit of R is then of maximal dimension, and because of Theorem II, we can apply Theorem I to Ω_R. As shown in [6], the set $S = S(K; R, e)$ of Chapter I, no. 2, can be identified with the set S' of line segments in $\mathfrak{t}$, obtained by choosing in each cell Δ of the fundamental chamber $\mathfrak{F}$ a suitable point, and connecting it to the origin 0 by a straight segment s. To Δ or to s, we associate the K-cycle of the geodesic segment $\exp \circ s$ (in the sense of Chapter I, no. 4), denoted by Γ_Δ, and the map of Γ_Δ into $\Omega(K)$ (using the identification of the various Ω_R above), denoted by f_Δ. The exceptional points of $\exp \circ s$ correspond to the points where s crosses singular planes in $\mathfrak{t}$ (we assume that s has no point of higher singularity). The exceptional stabilizers are then groups of the set $\mathfrak{K}$ of no. 3; the stabilizer of $\exp \circ s$ is T itself. It follows that all K-cycles are of the type discussed in no. 4. In more detail, we get from Theorem I the following proposition, describing the homology of $\Omega(K)$.

PROPOSITION 9.1. *For each cell Δ in $\mathfrak{F}$, let $P = \{p_1, \cdots, p_r\}$ be the ordered set of oriented planes in $\mathfrak{P}$ that the segment s from Δ to 0 crosses (in that order). The K-cycle Γ_Δ is then Γ_P, in the sense of* 4.1, *so that in particular* $\dim \Gamma_\Delta = 2r$. *The f_Δ-images of fundamental cycles γ_Δ of the Γ_Δ, with Δ ranging over the cells of $\mathfrak{F}$, form a basis for the (free) group $H_*(\Omega(K))$.*

It should be noted that strictly speaking the geodesic segment $\exp \circ s$ is not in the set S, but that a suitable inner automorphism of K will bring it into S. The abstract K-cycle will not change under this operation; and, K being connected, the image homology class in the loop space will also be the same (using again the identification of all the Ω_R).

For our application to homotopy, we need a partial determination of the cohomology ring of $\Omega(K)$. We quote some well known facts. Assume from now on K to be simple. The cohomology $H^*(K;\boldsymbol{Q})$ over the rationals $\boldsymbol{Q}$ is an exterior algebra with l generators of dimensions $2m_i-1$, with $2=m_1<m_2\leqq\cdots\leqq m_l$ (actually $m_2=m_3$ only for the type D_4). The cohomology $H^*(\Omega(K);\boldsymbol{Q})$ is a polynomial ring with l generators of dimensions $2m_i-2$. Since $H^*(\Omega(K))$ has no torsion, this implies that all $H^{2i+1}(\Omega(K))$ vanish, that $H^{2i}(\Omega(K))$ is infinite cyclic for $0\leqq i\leqq m_2-2$, and that $H^{2m_2-2}(\Omega(K))$ is free abelian of rank 2 or 3. Let η denote a generator of $H^2(\Omega(K))$; no power η^i vanishes. The second non-vanishing homotopy group of K (or $\Omega(K)$) depends on the divisibility properties (by integers) of the powers η^i, $2\leqq i\leqq m_2-1$, as described in the next proposition.

PROPOSITION 9.2. (A) *Suppose no power η^i, for $1\leqq i\leqq m_2-1$, is divisible by an integer >1; then $\pi_r(K)=0$ for $3<r<2m_2-1$, and $\pi_{2m_2-1}(K)$ is free abelian of rank one less than the number of m_i's equal to m_2;*

(B) *In the remaining case, let i_0 be the smallest exponent $i\leqq m_2-1$ for which η^i is divisible, and let η^{i_0} be divisible exactly by the positive integer q (>1); then $\pi_r(K)=0$ for $3<r<2i_0$, and $\pi_{2i_0}(K)=\mathbf{Z}_q$.*

Proof. The integral cohomology ring of the Eilenberg-MacLane space $K(\mathbf{Z},2)$, the infinite complex projective space, is a polynomial ring $\mathbf{Z}[x]$ with $\dim x=2$. By standard principles, there exists a map $\phi_\eta\colon \Omega(K)\to(\mathbf{Z},2)$ such that $\phi_\eta{}^*(x)=\eta$, and consequently $\phi_\eta{}^*(x^i)=\eta^i$. One can now easily determine the nature of the first non-vanishing relative homology group of ϕ_η (i.e. of the mapping cylinder of ϕ_η, with $\Omega(K)$ as the subset) in terms of the assumptions in (9.2); the Hurewicz isomorphism theorem and the known nature of the $\pi_r(K(\mathbf{Z},2))$ yield then (9.2). We see from (9.1) that η^i is divisible by q exactly if for every Δ with $\dim\Gamma_\Delta=2i$, the cocycle $f_\Delta{}^*(\eta^i)$ (a multiple of the fundamental cocycle of Γ_Δ) is divisible by q. We shall study this question using the description of Γ_Δ in (4.2), and begin by finding the expression for $f_\Delta{}^*(\eta)$.

10. The 3-spheres in K. We identify the Lie algebra of the subgroup S_1 of the group S_3 (cf. no. 3) with the real numbers $\boldsymbol{R}$ such that the exponential map is given by $\phi\to e^{i\phi}$, for $\phi\in\boldsymbol{R}$. There is then one root θ with $\theta(\phi)=\frac{1}{\pi}\phi$; the basic translation τ_θ is the number 2π; the system $\mathfrak{P}$ of oriented singular planes can be identified with the integral multiples of π; the fundamental

chamber is given by $\phi > 0$. Let n be any integer; let $s: [0,1] \to \boldsymbol{R}$ be a non-degenerate, linear map with $s(1) = 0$ such that for some $\bar{t} \in [0,1)$, we have $s(\bar{t}) = n \cdot \pi$, so that $\bar{t}$ is an exceptional value of the geodesic segment $\exp \circ s$ in S_3, according to Chapter I, no. 4. Put $R = \exp \circ s(0)$. We write $s = s' + s''$ with $s' = [0, \bar{t}]$ (possibly degenerate) and $s'' = [\bar{t}, 1]$. Generalizing slightly the construction of Chapter I, no. 5, we define a map $\bar{f}_n: S_3 \to \Omega(S_3; R, 1)$ by $\bar{f}_n(x) = \exp \circ s' + x \cdot \exp \circ s'' \cdot x^{-1}$. This is well defined at the junction point since the centralizer of $\exp \circ s(\bar{t})$ is S_3. As in Chapter I, no. 5, the map $\bar{f}_n$ is constant along the cosets of S_1, and induces a map $f_n: S_2 = S_3/S_1 \to \Omega(S_3; R, 1)$.

PROPOSITION 10.1. *Let y be the generator of $H_2(S_2)$ defined in* 3.3. *Then $\bar{y} = f_{1*}(y)$ is a generator of the infinite cyclic group $H_2(\Omega(S_3))$; and $f_{n*}(y) = n \cdot \bar{y}$.*

We identify here $\Omega(S_3; R, 1)$ with $\Omega(S_3) = \Omega(S_3; 1, 1)$; that $\bar{y}$ is a generator of $H_2(\Omega(S_3))$ is a special case of Proposition 9.1 for $K = S_3$, since the interval $(\pi, 2\pi)$ is the only cell Δ whose Γ_Δ has dimension 2, and since S_2 and f_1 are just the associated Γ_Δ and f_Δ.

To treat the factor n, we note first that we may choose s in such a way that $s(0) = 2n\pi$. Let ρ be the natural homomorphism of S_3 onto $S_3/\{1, -1\} = P_3$ (projective 3-space $= SO(3)$). Each $\rho \circ f_n(x)$, for $x \in S_2$, is then a loop in P_3 that goes through e at the parameter values $i/2n$, $0 \leqq i \leqq 2n$; in other words, it is a $2n$-fold Pontryagin product in $\Omega(P_3)$. Using the fact that the Pontryagin product in the loop space of a group is homotopy-commutative one verifies that $\rho \circ f_n$ is homotopic to the n-th Pontryagin power of $\rho \circ f_1$ (for $n \geqq 0$), and to the inverse of $\rho \circ f_1$ for $n = -1$. Since ρ induces a homeomorphism of $\Omega(S_3)$ with the e-component of $\Omega(P_3)$, and since y, being spherical, is primitive, one concludes now easily that $f_{n*}(y) = n \cdot \bar{y}$.

For any root $\theta \in \mathcal{R}$, let $g_\theta: S_3 \to K$ be the composition of the map β_θ of no. 3 with the inclusion of K_θ in K, and let $\bar{g}_\theta$ be the induced map of the loop spaces.

PROPOSITION 10.2. (A) *The element $\bar{g}_{\mu*}(\bar{y}) = y_K$ (where μ is the dominant root and $\bar{y}$ is defined in* 10.1) *is a generator of $H_2(\Omega(K))$.*

(B) *For any root $\theta \in \mathcal{R}$, the following relation holds:*

$$\bar{g}_{\theta*}(\bar{y}) = ((\tau_\theta, \tau_\theta)/(\tau_\mu \cdot \tau_\mu)) \cdot y_K.$$

(Note that the factor of y_K is always an integer.)

For part (A), we use (9.1). There is exactly one cell Δ in $\mathcal{F}$ with

dim $\Gamma_\Delta = 2$, namely the cell obtained by reflecting the fundamental cell Δ_y across the plane $(\mu, 1)$; we have $\Gamma_\Delta = K_\mu/T$. One verifies, by considering the definition of f_Δ, Chapter I, no. 5, the relation

$$f_\Delta \circ \bar{\chi}_1 = \bar{g}_\mu \circ f_1, \tag{10.3}$$

with f_1 as defined above, and $\bar{\chi}_1 : S_2 \to \Gamma_\Delta$ defined in no. 4. [Actually, this equation has to be understood as a homotopy. The map f_Δ is defined by a certain segment s in t. As far as $f_\Delta \circ \bar{\chi}_1$ goes, one can replace s by any segment in t that has 0 as end point and for whose initial point the inequality $\mu(s(0)) \geqq 1$ holds, provided that one disregards all exceptional points of s except the one where s meets the plane $(\mu, 1)$. In terms of Chapter I, no. 5, this means that instead of W_n we act on s only with $T \times \cdots \times T \times K_n$. It is now possible to take for s the image, under the differential $\dot{g}_\mu$ of g_μ, of the segment used for the construction of f_1 in the Lie algebra $\boldsymbol{R}$ of S_1; this produces actual equality in the above relation.] It follows that $f_{\Delta *}(y_1) = \bar{g}_{\mu *}(\bar{y})$; since y_1 is the fundamental cycle of K_μ/T, part (A) follows.

The standard transgression argument shows that, with z denoting a generator of $H_3(S_3)$, the element $g_{\mu *}(z)$ is a generator of $H_3(K)$. We shall prove (10.2 (B)) by showing that $g_{\theta *}(z) = ((\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu)) \cdot g_{\mu *}(z)$.

Let ψ_K be the Cartan-form of K, i.e. the closed invariant differential 3-form defined, at e, by $\psi_K(X, Y, Z) = (X, [Y, Z])$. Let τ, τ', τ'' be a basis for the Lie algebra of S_3, with τ the basic translation in the Lie algebra of S_1, and $[\tau', \tau''] = \tau$. By (3.4), we have $\beta_\theta(\tau) = \tau_\theta$, and therefore $g_\theta{}^* \psi_K(\tau, \tau', \tau'') = (\tau_\theta, \tau_\theta)$. This implies $g_\theta{}^* \psi_K = ((\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu)) \cdot g_\mu{}^* \psi_K$, and (10.2 (B)) follows.

11. Computation of $f_\Delta{}^*(\eta)$. Let the generator η of $H^2(\Omega(K))$ be so chosen that $\eta(y_K) = 1$, with y_K as in (10.2(A)). Let Δ be any cell in $\mathfrak{F}$, and let $P = \{p_1, \cdots, p_r\}$, Γ_Δ and f_Δ be as in (9.1). By no. 4, to each $p_i \in P$, there corresponds a basis element y_i of $H_2(\Gamma_\Delta)$; the determination of $f_\Delta{}^*(\eta) \in H^2(\Gamma_\Delta)$ is equivalent to that of $f_{\Delta *}(y_i)$, $1 \leqq i \leqq r$.

PROPOSITION 11.1.

$$f_{\Delta *}(y_i) = n_i((\tau_i, \tau_i)/(\tau_\mu, \tau_\mu)) \cdot y_K,$$

with $\tau_i = \tau_{p_i}$ and $n_i = n_{p_i}$, the translation and the multiplicity associated with the singular plane p_i.

For the proof, one verifies, as in the proof of (10.2), that the commutativity relation $f_\Delta \circ \bar{\chi}_i = \bar{g}_i \circ f_{n_i}$ holds, where f_{n_i} is as in (10.1), and where $\bar{g}_i$

is the map of the loop spaces induced by the map $g_i\colon S_3 \to K_i \subset K$. (11.1) follows then from (10.1) and (10.2 (B)). By simple duality, we have

PROPOSITION 11.2.

$$f_\Delta^*(\eta) = (\tau_\mu, \tau_\mu)^{-1} \sum_1^r n_i(\tau_i, \tau_i) x_i;$$

here τ_i, n_i are the basic translation and the multiplicity associated with the i-th singular plane p_i along the segment s from Δ to 0.

12. Computation of $\pi_4(K)$. To determine $\pi_4(K)$, we have to consider the element η^2, according to (9.2); according to (9.1), this amounts to considering all cells Δ with $\dim \Gamma_\Delta = 4$. Assume that the rank l of K is > 1. Let Δ_1 be the cell obtained from the fundamental cell $\Delta_{\mathfrak{F}}$ by reflection across the plane $(\mu, 1)$. It is clear that the cells that correspond to dimension 4 are the cells in $\mathfrak{T}$ adjoining to Δ_1 (different from $\Delta_{\mathfrak{F}}$). Let Δ be such a cell (actually there are one or two such, with two occurring for the type A_n, $n \geqq 2$, only). The segments from Δ to 0 has 2 exceptional points; the first belongs to a certain root, say θ; the second belongs to the dominant root. By (4.2), the ring $H^*(\Gamma_\Delta)$ is then generated by two variables x_1, x_2 subject to ${x_1}^2 = 0$ and ${x_2}^2 = -ax_1x_2$, where $a = \mu(\tau_\theta)$. The multiplicities n_1, n_2 of (11.2) are both 1, and we have $f_\Delta^*(\eta) = ((\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu)) \cdot x_1 + x_2$. Squaring, we obtain

$$f_\Delta^*(\eta^2) = (2(\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu) - a)x_1x_2.$$

Since τ_μ belongs to the closure of $\mathfrak{T}$, we have $\theta(\tau_\mu) > 0$ ($\neq 0$, since τ_μ and τ_θ are clearly not orthogonal). Moreover, since $(\tau_\mu, \tau_\mu) \leqq (\tau_\theta, \tau_\theta)$ (τ_μ is a "shortest" translation), we have $\theta(\tau_\mu) < \theta(\tau_\theta) = 2$, and so $\theta(\tau_\mu) = 1$. It follows that $(\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu) = \mu(\tau_\theta)/\theta(\tau_\mu) = \mu(\tau_\theta)$, and so

$$f_\Delta^*(\eta^2) = \mu(\tau_\theta)x_1x_2 \tag{12.1}$$

It follows from (12.1) that η^2 is divisible by the integer $q \geqq 1$ if and only if the Cartan integers $\mu(\tau_\theta)$, for the θ's involved, are so divisible. Recalling that these roots θ are obtained by reflecting across the plane $(\mu, 1)$ those fundamental roots that are not orthogonal to μ, and recalling that the translations belonging to the fundamental roots generate the lattice $\mathfrak{T}$, one sees easily that η^2 is divisible by q exactly if all values $\mu(X)$, for $X \in \mathfrak{T}$, are so divisible. Since $\mu(\tau_\mu) = 2$, this means $q = 1$ or 2. Moreover, q will be 1 exactly if there exists an $X \in \mathfrak{T}$ with $\mu(X) = 1$. This proves Theorem IV(a). The statements of IV(b) are obtained by a simple inspection of the diagrams

of the simple groups. The case $l=1$, i.e. the group S_3 itself, can be treated with the same method; there is only one root, the multiplicity n_1 is 2, as is the Cartan integer a; n_2 is 1; $f_\Delta^*(\eta)=2x_1+x_2$; $f_\Delta^*(\eta^2)=2x_1x_2$. The result $\pi_4(S_3)=\boldsymbol{Z}_2$ is of course well known

13. Homotopy groups of E_6, E_7, E_8. To have a way of picturing the arrangement of the cells in the fundamental chamber, we associate to the Lie group K a graph $\Phi=\Phi(K)$, the 1-skeleton of the dual of the triangulation of $\mathfrak{F}$ by the cells, with labelled edges. To each cell Δ in $\mathfrak{F}$, there is associated a vertex v_Δ, and to any $(l-1)$-face between two cells is associated an edge connecting the two vertices; the edge is labelled with the plane $p\in\mathfrak{P}$ in which the $(l-1)$-face lies. There is a distinguished vertex $v_{\mathfrak{F}}$, corresponding to the lowest cell $\Delta_{\mathfrak{F}}$. We define a function d of the vertices by setting $d(v)=$ smallest number of edges in a path from v to $v_{\mathfrak{F}}$. One verifies that for any cell Δ with segment s from Δ to 0 and K-cycle Γ_Δ, as in (9.1), one has $\dim\Gamma_\Delta=2d(v_\Delta)$, and that the singular planes occur along s in the same order as along a *suitable* minimal path from v_Δ to $v_{\mathfrak{F}}$ in Φ. We write Φ_i for the subgraph of Φ containing all vertices with d-values $\leqq i$.

Then, for any K $(\neq D_4)$, the graph Φ_{m_2-1} (here m_2 is the second primitive exponent of K, cf. no. 9) has the form

Figure 2.

as follows at once from the behavior of the Betti numbers. Since for all the vertices shown, the minimal path to $v_{\mathfrak{F}}$ is unique, one can read off the nature of the K-cycles Γ_Δ, provided one knows which planes of $\mathfrak{P}$ are attached to the various edges. The first edge (i.e., the edge at $v_{\mathfrak{F}}$) is of course labelled with the dominant root μ, or better, with the plane $(\mu,1)$. We now describe the labelling of Φ_{m_2-1} for the exceptional groups E_6, E_7, E_8, in terms of the usual fundamental roots $\{\phi_i\}_1^l$ (cf. the Coxeter (Schläfli) diagrams below). To have a short description, we state first that all the planes p occurring are of the form $(\theta,1)$, i.e. they have $n_p=1$. Further, it turns out that for each edge after the first one, the root attached is obtained by subtracting a suitable fundamental root from the root attached to the edge immediately preceding in Φ. We give below the system consisting of the dominant root and the fundamental roots to be subtracted successively; the last two entries correspond to the two edges leading to v' and v''.

$$
\begin{aligned}
&E_6\colon \{\mu, \phi_6, \phi_3; \phi_2, \phi_4\}\\
(13.1)\qquad &E_7\colon \{\mu, \phi_6, \phi_5, \phi_4; \phi_3, \phi_7\}\\
&E_8\colon \{\mu, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5; \phi_6, \phi_8\}.
\end{aligned}
$$

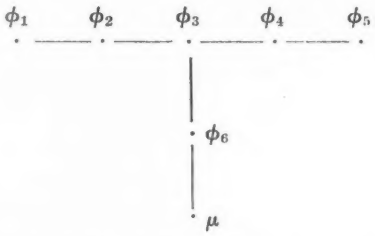
The singular plane attached to the 3rd edge of $\Phi_4(E_6)$, e.g., is then $(\mu - \phi_6 - \phi_3, 1)$; the plane attached to the edge leading to v' is $(\mu - \phi_6 - \phi_3 - \phi_2, 1)$. To prove that these are the correct values, we proceed as follows:

(A) The fundamental roots indicated in (13.1), are the only ones at each stage in the processe that can be subtracted from the root obtained up to the point in question and yield again a root as difference. To verify this we start from the remark that for any two roots θ, θ' of a group K, the difference $\theta - \theta'$ is again a root if $\theta'(\tau_\theta)$ is positive (cf. [5]); moreover, for E_6, E_7, E_8 the reverse implication is also true, since all basic translations are of the same length, say 1. Further, we have for the fundamental roots ϕ_i and the corresponding basic translations λ_i the usual relations

$$
\begin{aligned}
&\phi_i(\lambda_i) = 2,\\
(13.2)\qquad &\phi_i(\lambda_j) = 2(\lambda_i, \lambda_j) = -1, \text{ if } \phi_i \text{ and } \phi_j \text{ are adjacent in the Coxeter diagram,}\\
&\phi_i(\lambda_j) = 0 \text{ otherwise.}
\end{aligned}
$$

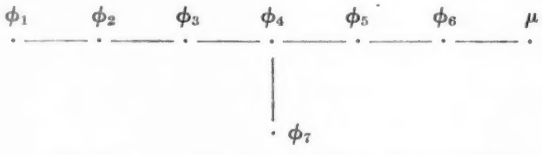
One verifies that in our three cases the dominant root μ has $\mu(\lambda_i)$ positive for exactly one of the λ_i, and 0 for all others, so that exactly one $\mu - \phi_i$ is a root (cf. also [5]). We reproduce the three Coxeter diagrams, with in each case a new vertex, corresponding to the dominant root, joined by an edge to the unique vertex whose corresponding λ_i is not orthogonal to τ_μ.

E_6:



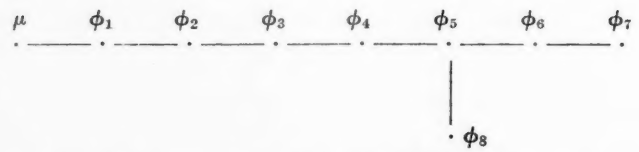
$$\mu = \phi_1 + 2\phi_2 + 3\phi_3 + 2\phi_4 + \phi_5 + 2\phi_6,$$

E_7:



$$\mu = \phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 + 3\phi_5 + 2\phi_6 + 2\phi_7,$$

E_8:



$$\mu = 2\phi_1 + 3\phi_2 + 4\phi_3 + 5\phi_4 + 6\phi_5 + 4\phi_6 + 2\phi_7 + 3\phi_8.$$

As an example, we compute, for E_6, $\mu(\lambda_1) = \phi_1(\lambda_1) + 2\phi_2(\lambda_1) = 2 - (2 \cdot -1) = 0$ and $\mu(\lambda_6) = 3\phi_3(\lambda_6) + 2\phi_6(\lambda_6) = 3 \cdot -1 + 2.2 = 1$. One verifies the statement made in (A) about the roots in 13.1 directly from the diagrams. As an example, for E_6, to find a ϕ_i for which $\mu - \phi_6 - \phi_i$ is a root, we have to find a ϕ_i for which $\phi_i(\tau_\mu - \lambda_6) = \phi_i(\tau_\mu) - \phi_i(\lambda_6) > 0$. Clearly i must be different from 6; but then $\phi_i(\tau_\mu) = 0$, and the only root with $\phi_i(\lambda_6) \neq 0$ (and actually $= -1$ by (13.2)) is the root ϕ_3, adjacent to ϕ_6.

(B) If for two planes $p = (\theta, n)$, $p' = (\theta', n')$ of $\mathfrak{P}$, the linear function $\theta/n - \theta'/n'$ is positive on $\mathfrak{F}$, then clearly the first (i.e. nearest to $v_{\mathfrak{F}}$) occurrence of p at an edge of Φ comes before any occurrence of p'. One verifies now that the roots θ constructed by the prescription in (13.1), are successively smaller in this sense, since each time a fundamental root is subtracted (except that the last two roots are not comparable in this ordering), and that they are all greater than $\frac{1}{2} \cdot \mu$ (by explicit computation; e.g., for E_6, $\mu - \phi_6 - \phi_3 - \phi_2 = \phi_1 + \phi_2 + \phi_3 + 2\phi_4 + \phi_5 + \phi_6 > \frac{1}{2} \cdot \mu$ on $\mathfrak{F}$).

It is clear from (A) and (B) that the labelling of Φ_{m_2-1}, described by (13.1), is the correct one.

For each of E_6, E_7, E_8, we shall now compute the cohomology ring of the manifold Γ', attached by (9.1) to the vertex v' of the graph Φ_{m_2-1}, and also the image, under the associated map f', of η (by (11.2)) and of the relevant power of η; using (9.2), this will yield Theorem V(a), (b), (c). If s is a segment from the cell Δ', corresponding to v' in figure 2, to 0, then the singular planes crossed by s are identical with the singular planes attached to the edges of Φ_{m_2-1}, moving from v' to $v_{\mathfrak{F}}$. All the multiplicities are 1, as noted before; all the "weights" $(\tau_i, \tau_i)/(\tau_\mu, \tau_\mu)$ occurring in (11.2) are also 1. The roots along s are given by the description in (13.1) (but in reverse order!). For E_6, for instance, the singular planes p_1, p_2, p_3, p_4 are given by $(\mu - \phi_6 - \phi_3 - \phi_2, 1)$, $(\mu - \phi_6 - \phi_3, 1)$ $(\mu - \phi_6, 1)$ $(\mu, 1)$.

It turns out that all the Cartan integers, appearing in the description (4.2) of the cohomology of our Γ's are 1. As an example, for E_6, the integer $\mu(\tau_\mu - \lambda_6) = \mu(\tau_\mu) - \mu(\lambda_6) = 2 - 1 = 1$. We omit the details.

For the Γ' of E_6, the cohomology is then generated by variables x_1, x_2, x_3, x_4, with the relations

$$
\begin{aligned}
&x_1^2 = 0,\\
&x_2(x_1 + x_2) = 0,\\
&x_3(x_1 + x_2 + x_3) = 0,\\
&x_4(x_1 + x_2 + x_3 + x_4) = 0,
\end{aligned}
\tag{13.1}
$$

and $f'^*(\eta)$ is given by $x_1 + x_2 + x_3 + x_4$. The fundamental cocycle of Γ' is given by $x_1x_2x_3x_4$; we claim

$$f'^*(\eta^4) = x_1x_2x_3x_4. \tag{13.4}$$

For the (elementary) computation, we introduce a new basis by setting $z_1 = x_1$, $z_1 = x_1 + x_2$, $z_3 = x_1 + x_2 + x_3$, $z_4 = x_1 + x_2 + x_3 + x_4$. The relations turn into

$$
\begin{aligned}
&z_1^2 = 0,\ z_2^2 = z_1z_2,\ z_3^2 = z_2z_3,\ z_4^2 = z_3z_4,\\
&\text{and } f'^*(\eta) \text{ is given by } z_4.
\end{aligned}
\tag{13.5}
$$

Now $(z_4)^4 = (z_3z_4)^2 = z_2z_3^2 \cdot z_4 = z_2^2 \cdot z_3z_4 = z_1z_2z_3z_4 = x_1x_2x_3x_4$, q. e. d.

It follows that the powers η^i, $1 \leqq i \leqq 4$, are not divisible by any integers $q > 1$, and Theorem V(a) follows from (9.2).

The computations for E_7 and E_8, with 5, resp. 7 variables x_i are entirely analogous.

14. $\pi_{10}(G_2) \otimes \mathbf{Z}_3$, preparation. Following a suggestion of J. C. Moore and A. S. Shapiro, we shall compute $\pi_{10}(G_2) \otimes \mathbf{Z}_3$ by considering, in addition to the ring structure of $H^*(\Omega(G_2); \mathbf{Z}_3)$ up to dimension 10, a certain Steenrod power (cf. [19] for this concept).

We begin by listing the necessary facts about G_2 (in a notation somewhat different from no. 7).

(1) Fundamental roots ϕ_1, ϕ_2; corresponding translations λ_1, λ_2;

(2) Coxeter diagram $\overset{\phi_1}{\circ}\!\!=\!\!\!=\!\!\overset{\phi_2}{\circ}$; $(\lambda_1, \lambda_1) = 3$, $(\lambda_2, \lambda_2) = 1$, $(\lambda_1, \lambda_2) = -\frac{3}{2}$;

(3) Fundamental Cartan integers: $\phi_1(\lambda_1) = \phi_2(\lambda_2) = 2$, $\phi_1(\lambda_2) = -1$, $\phi_2(\lambda_1) = -3$;

(4) Positive roots (in lexicographic order): $3\phi_1 + 2\phi_2$ (dominant),

$$3\phi_1 + \phi_2, 2\phi_1 + \phi_2, \phi_1 + \phi_2, \phi_1, \phi_2;$$

corresponding translations:

$$\lambda_1 + 2\lambda_2, \lambda_1 + \lambda_2, 2\lambda_1 + 3\lambda_2, \lambda_1 + 3\lambda_2, \lambda_1, \lambda_2;$$

norm-squares of these translations: $1, 1, 3, 3, 3, 1$;

(5) $$m_2 = 5;$$

graph Φ_5:

singular planes p attached to the edges of Φ_5 (in order): $(3\phi_1 + 2\phi_2, 1)$, $(3\phi_1 + \phi_2, 1)$, $(2\phi_1 + \phi_2, 1)$, $(3\phi_1 + 2\phi_2, 2)$; $(3\phi_1 + \phi_2, 2)$ (to v'), $(\phi_1 + \phi_2, 1)$ (to v'');

beginning of the fundamental chamber:

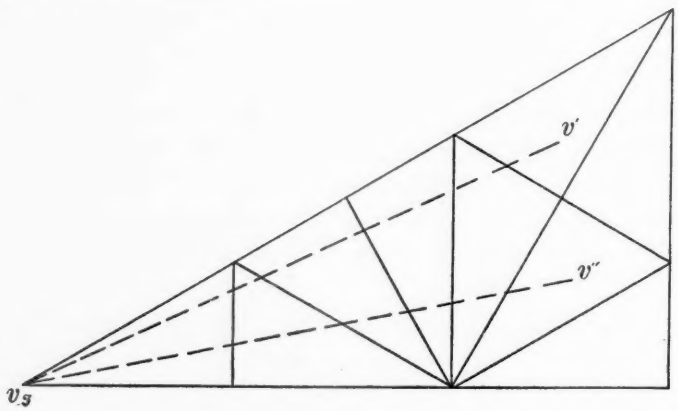


Figure 3.

As before in nos. 9 and 11, we denote by η the generator of $H^2(\Omega(G_2))$, defined by $\eta(y_{G_2}) = 1$. Our first aim is the following partial description of $H^*(\Omega(G_2); \mathbf{Z}_3)$.

PROPOSITION 14.1. *The element η^3 of $H^6(\Omega(G_2))$ is divisible exactly by 3. Put $u = \eta^3/3$. Then a basis for $\sum_0^{10} H^i(\Omega(G_2); \mathbf{Z}_3)$ is given by $\{1; \eta; \eta^2; u; u\cdot\eta; u\cdot\eta^2, P^1u\}$; here the image of η etc. under the coefficient reduction $\mathbf{Z} \to \mathbf{Z}_3$ is again denoted by η etc.; and P^1 is the Steenrod power that send $H^i(X; \mathbf{Z}_3)$ into $H^{i+4}(X; \mathbf{Z}_3)$.*

Remark. The fact that η^3 is divisible exactly by 3, is responsible for the complexity of the problem of finding $\pi_{10}(G_2) \otimes \mathbf{Z}_3$. From now on we write Ω for $\Omega(G_2)$. We shall prove Proposition 14.1 by considering the two G_2-cycles Γ', Γ'' attached, by (9.1), to the cells Δ', Δ''. [A part of the result.

in fact all except the determination of P^1u, follows also, for instance, from the fact that, G_2 having only two-torsion, $H^*(\Omega;\mathbf{Z}_3)$ is a divided polynomial ring.] We denote by f', f'' the maps of Γ', Γ'' into Ω, defined in Chapter I, (5.4).

Γ': The variables $x_1,\cdots,x_5$, generating $H^*(\Gamma')$, correspond, in order, to the roots $3\phi_1+\phi_2$, $3\phi_1+2\phi_2$, $2\phi_1+\phi_2$, $3\phi_1+\phi_2$, $3\phi_1+2\phi_2$. The relations are given by

$$\begin{aligned} &x_1^2=0,\\ &x_2(x_2+x_1)=0,\\ &x_3(x_3+x_2+x_1)=0,\\ &x_4(x_4+3x_3+x_2+2x_1)=0,\\ &x_5(x_5+x_4+3x_3+2x_2+x_1)=0. \end{aligned} \tag{14.2}$$

As an example, $a_{43}=(3\phi_1+\phi_2)(2\lambda_1+3\lambda_2)=6\phi_1(\lambda_1)+9\phi_1(\lambda_2)+2\phi_1(\lambda_1)+3\phi_2(\lambda_2)=12-9-6+6=3$. The multiplicities n_i of 11.2 are given by $2,2,1,1,1$, and the weights $(\tau_i,\tau_i)/(\tau_\mu,\tau_\mu)$ by $1,1,3,1,1$, so that

$$a'=f'^*(\eta)=2x_1+2x_2+3x_3+x_4+x_5.$$

The following computations are facilitated by making the change of basis

$$\begin{aligned} a'&=2x_1+2x_2+3x_3+x_4+x_5,\\ b'&=2x_1+2x_2+3x_3+x_4,\\ c'&=\qquad\qquad\quad x_3,\\ d'&=x_1+x_2,\\ e'&=x_1. \end{aligned}$$

This transforms the relations into

$$\begin{aligned} a'^2&=a'b'+a'e'-b'e',\\ b'^2&=3(b'c'+b'd'-c'd'+c'e')-b'e',\\ c'^2&=-c'd',\\ d'^2&=d'e',\\ e'^2&=0. \end{aligned} \tag{14.3}$$

One clearly has $a'b'c'd'e'=x_1x_2x_3x_4x_5$. One computes

$$(a')^3=3(a'-e')(b'c'+b'd'-c'd'+c'e') \tag{14.4}$$

and, with $u'=(a')^3/3$,

$$u'(a')^2=4a'b'c'd'e'. \tag{14.5}$$

Γ'': The variables $x_1, \cdots, x_5$ in $H^2(\Gamma'')$ correspond to the roots $\phi_1+\phi_2$, $3\phi_1+2\phi_2$, $2\phi_1+\phi_2$, $3\phi_1+\phi_2$, $3\phi_1+2\phi_2$. The relations are computed to be

$$
\begin{aligned}
&x_1^2 = 0,\\
&x_2(x_2+3x_1) = 0,\\
&x_3(x_3+x_2+x_1) = 0,\\
&x_4(x_4+3x_3+2x_2) = 0,\\
&x_5(x_5+x_4+3x_3+2x_2+3x_1) = 0.
\end{aligned}
\tag{14.6}
$$

The multiplicities n_i of are $1,2,1,1,1$; the weights $(\tau_i,\tau_i)/(\tau_\mu,\tau_\mu)$ are $3,1,3,1,1$, so that

$$a'' = f''^*(\eta) = 3x_1+2x_2+3x_3+x_4+x_5.$$

We change variables according to

$$
\begin{aligned}
a'' &= 3x_1+2x_2+3x_3+x_4+x_5,\\
b'' &= 3x_1+2x_2+3x_3+x_4,\\
c'' &= x_1 \qquad\quad + x_3,\\
d'' &= \qquad x_2,\\
e'' &= x_1.
\end{aligned}
$$

The relations go over into

$$
\begin{aligned}
&a''^2 = a''b'',\\
&b''^2 = 3(b''c''+b''e''+b''d''-c''d''-3c''e''),\\
&c''^2 = -c''d''+c''e''+d''e'',\\
&d''^2 = -3d''e'',\\
&e''^2 = 0.
\end{aligned}
\tag{14.7}
$$

One computes that

$$a''^3 = 3a''(b''c''+b''e''+b''d''-c''d''-3c''e''). \tag{14.8}$$

From (14.4) and (14.8), we conclude, by (9.2), that η^3 is divisible exactly by 3; let $\eta^3/3 = u$, and $a''^3/3 = u''$. We reduce now the coefficients in $H^*(\Gamma'')$ from $\boldsymbol{Z}$ to $\boldsymbol{Z}_3$. We have then

$$u''a''^2 = (a'')^3(b''c''+\cdots) = 0 \text{ in } H^*(\Gamma''; \boldsymbol{Z}_3), \tag{14.9}$$

since $(a'')^3$ is zero mod 3.

Utilizing the fact that the Steenrod power P^1 is a derivation (Cartan formula), and that P^1 applied to a two-dimensional element yields the third power of the element, we obtain after a short computation

$$P^1u'' = -a''b''c''d''e''. \tag{14.10}$$

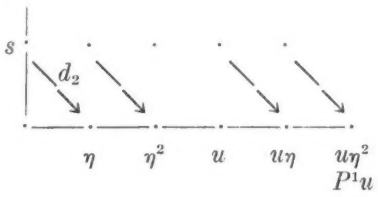
From (14.5) and (14.10), we conclude that $u \cdot \eta^2$ and P^1u are non-zero elements of $H^{10}(\Omega; \mathbf{Z}_3)$, since they map into non-zero elements under f', resp. f''. From (14.9) and (14.10), we conclude, applying f'', that $u \cdot \eta^2$ and P^1u are linearly independent. Since $H^{10}(\Omega; \mathbf{Z}_3) \approx \mathbf{Z}_3 + \mathbf{Z}_3$, this proves (14.1).

15. The 3-component of $\pi_{10}(G_2)$. We compute $\pi_i(\Omega) \otimes \mathbf{Z}_3$, $i \leqq 9$, by a straightforward application of the method of killing homotopy groups, the principles of which we assume known. Let β denote the Bokstein operator $\beta: H^i(X; \mathbf{Z}_3) \to H^{i+1}(X; \mathbf{Z}_3)$, derived from the coefficient sequence

$$0 \to \mathbf{Z}_3 \to \mathbf{Z}_9 \to \mathbf{Z}_3 \to 0.$$

Throughout this section, $H^i(X)$ will mean $H^i(X; \mathbf{Z}_3)$. We utilize known facts about the cohomology of the Eilenberg-MacLane spaces $K(\mathbf{Z}_3, n)$, including the behavior of the Bokstein operator.

We kill $\pi_2(\Omega)$ by a suitable bundle X_1, over Ω, with $K(\mathbf{Z}, 1)$ or S_1 (the circle) as a fiber. The E_2-term of the corresponding spectral sequence is $H^*(\Omega) \otimes H^*(S_1)$. Let s denote the generator of $H^1(S_1)$ that maps into η under d_2. We have then $d_2(\eta \otimes s) = \eta^2$, $d_2(\eta^2 \otimes s) = 0$, $d_2(u \otimes s) = u \cdot \eta$, $d_2(u \cdot \eta \otimes s) = u \cdot \eta^2$.



It follows that $H^*(X_1)$, up to dimension 10, has the set $\{1, x, v, P^1v\}$ as basis, where $\dim x = 5$, and where v is the image of the element u of $H^6(\Omega)$ under the projection $X_1 \to \Omega$.

Furthermore the relation

$$\beta(x) = \pm v$$

holds. This follows from the fact that, using the integers as coefficients in the above computation, we have $d_2(\eta^2 \cdot s) = 3u$, so that the integral element v', whose reduction mod 3 yields v, satisfies $3v' = 0$.

We now kill x by constructing a suitable bundle X_2 over X_1, with $K(\mathbf{Z}_3, 4)$ as fiber. Let y be the generating class of $K(\mathbf{Z}_3, 4)$; the cohomology of $K(\mathbf{Z}_3, 4)$,

up to dimension 9, is given by the basis $\{1, y, \beta y, y^2, P^1y, \beta P^1y, P^1\beta y, y \cdot \beta y\}$. In the spectral sequence of $X_2 \to X_1$, we have $E_2 = H^*(X_1) \otimes H^*(K(\mathbf{Z}_3, 4)) = E_5$, and $d_5(y) = x$. It follows that

$$d_6(\beta y) = v, d_5(y^2) = 2x \otimes y, d_5(P^1y) = \cdots = d_9(P^1y) = 0,$$
$$d_5(y \cdot \beta y) = x \otimes \beta y, d_{10}(P^1\beta y) = P^1v, d_{10}(\beta P^1y) = 0.$$

(Note that βy and P^1y are transgressive, since y is.)

One computes thus that in E_∞ the only elements of dimension $\leqq 9$ are the linear combinations of the elements $1, \bar{w}, \bar{z}$, which come from $1, P^1y, \beta P^1y$ in E_2. It follows that $H^*(X_2)$, up to dimension 9, has a basis $\{1, w, z\}$, with $\dim w = 8$, $\dim z = 9$. Moreover, $\beta w = z$, since w and z map into P^1y, βP^1y under the inclusion of the fiber $K(\mathbf{Z}_3, 4)$ in X_2.

We now kill w in $H^*(X_2)$ by constructing a suitable bundle X_3 over X_2, with fiber $K(\mathbf{Z}_3, 7)$. If t is the generating class of $H^7(\mathbf{Z}_3, 7)$, then βt generates $H^8(\mathbf{Z}, 7)$. In the spectral sequence we conclude from $d_8 t = w$ and $d_9(\beta t) = \beta w = z$ that $H^i(X_3) = 0$ for $0 < i \leqq 9$.

From the values of the first non-trivial cohomology groups (in positive dimensions) of Ω and the X_i, $i = 1, 2, 3$,, we read off:

$$\pi_i(\Omega) \otimes \mathbf{Z}_3 = \mathbf{Z}_3 \text{ for } i = 2, 5, 8,$$
$$= 0 \text{ for the other } i\text{-values} \leqq 9.$$

Since $\pi_i(\Omega) \approx \pi_{i+1}(G_2)$, this implies in particular $\pi_{10}(G_2) \otimes \mathbf{Z}_3 = 0$, and Theorem V(d) is proved.

Chapter IV. Applications to Symmetric Spaces.

1. Introduction. In Chapter II, three actions were shown to be variationally complete for a symmetric pair (G, K): a) $K \times K$ on G, b) K on G/K, c) K on $\mathfrak{p}$ (by the adjoint action). In this chapter, we study some immediate consequences of Theorem I for these situations. Actually the cases a) and b) are equivalent in the following sense (this has appeared already in Chapter II, no. 6): Let $p: G \to G/K$ be the natural projection. If N' is a $K \times K$-orbit in G, then $N = p(N')$ is a K-orbit in G/K. Let R' be a point in G, and put $R = p(R')$. The map p induces a map p' of the path space $\Omega(G; R', N')$ (cf. Chapter I, no. 2) into $\Omega(G/K; R, N)$. It follows easily from the covering homotopy theorem for the fibering $G \to G/K$, that p' induces an isomorphism of the homotopy and therefore also of the homology of the two spaces.

By Theorem I, the mod 2 homology of $\Omega(G;R',N')$ is completely determined by the set $S=S(G;R',N')$ of transversal geodesics from R' to N' in G, as graded by the index δ_s. Our aim is to describe S and δ in each case in terms of invariants of (G,K). As we will show, the diagram of the symmetric pair (G,K) in the sense of E. Cartan [9], a variant of the diagram of a Lie group, contains all the pertinent information. Since there is no description of this diagram in the current literature, we reprove certain of Cartan's results. In particular, we have included Hunt's version of the proof of the conjugacy of the maximal tori of a symmetric space.

Properly understood, all the notions associated with the diagram of a Lie group generalize to the diagram of a symmetric space.

We will start with the adjoint action of K on $\mathfrak{p}$. Here the space $\Omega(\mathfrak{p};R,N)$ is homotopic to the orbit N. Our formulas therefore describe the homology of the orbits of K in $\mathfrak{p}$.

2. The adjoint action of K on $\mathfrak{p}$. We recall the pertinent definitions from Chapter II. The $(+1)$-eigen space of $*$ is the Lie algebra $\mathfrak{k}$ of K; its orthogonal complement, the (-1)-space of $*$, is $\mathfrak{p}$. Thus $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. For any $A\in\mathfrak{g}$, we have $\mathfrak{g}_A=\{X:[A,X]=0\}$; $\mathfrak{p}_A=\mathfrak{g}_A\cap\mathfrak{p}$; $\mathfrak{k}_A=\mathfrak{g}_A\cap\mathfrak{k}$.

PROPOSITION 2.1. *For any $A\in\mathfrak{p}$, the equations*

$$\mathfrak{g}_A=\mathfrak{k}_A\oplus\mathfrak{p}_A \tag{a}$$

$$\mathfrak{p}=[\mathfrak{k},A]\oplus\mathfrak{p}_A \tag{b}$$

represent orthogonal decompositions.

Relation (2.1(a)) follows from Chapter V, 3.1; for (2.1(b)), we use Chapter II, (7.5), with A replaced by zero and X replaced by A.

Let $\mathfrak{g}_A{}^{\perp}$ denote the orthogonal complement of $\mathfrak{g}_A$ in $\mathfrak{g}$. We see from (2.1(a)) and (2.1(b)) that $[\mathfrak{k},A]=\mathfrak{g}_A{}^{\perp}\cap\mathfrak{p}$, and that $\mathrm{ad}A$ maps $\mathfrak{g}_A{}^{\perp}\cap k$ isomorphically onto $\mathfrak{g}_A{}^{\perp}\cap\mathfrak{p}$, so that these two spaces have the same dimension. We conclude that

$$\dim\mathfrak{k}_A-\dim\mathfrak{p}_A=\dim\mathfrak{t}-\dim\mathfrak{p}, \tag{2.2}$$

independent of A.

This can also be expressed by saying that the trace of $*$ on $\mathfrak{g}_A$ equals the trace of $*$ on $\mathfrak{g}$, for any A.

DEFINITION 2.3. *A maximal abelian subalgebra of $\mathfrak{p}$ is called a Cartan algebra of (G,K). If $\mathfrak{t}$ is such an algebra, $A\in\mathfrak{t}$ is called a general point of $\mathfrak{t}$ if $\mathfrak{p}_A=\mathfrak{t}$. The dimension of $\mathfrak{t}$ is called the rank of (G,K).*

It is clear from general theory that Cartan algebras as well as general points on them exist. In the following, we *denote by* $\mathfrak{t}$ *a fixed Cartan algebra of* (G,K), *and let* A *be a fixed general point of* $\mathfrak{t}$.

Let Q be a point of $\mathfrak{p}$, and let N be the K-orbit of Q. As in Chapter I, no. 2, we denote by $S(\mathfrak{p};A,N)$ the set of geodesics (i.e. straight line segments in $\mathfrak{p}$) which are transversal to the action of K on $\mathfrak{p}$ and join A to points of N. Let $S(\mathfrak{t};A,\mathfrak{t}\cap N)$ be the set of straight lines in $\mathfrak{t}$ joining A to points of $\mathfrak{t}\cap N$.

PROPOSITION 2.4. $S(\mathfrak{p};A,N)=S(\mathfrak{t};A,\mathfrak{t}\cap N)$.

Proof. A transversal line from A to N must be perpendicular to the orbit of A at A (cf. Chapter I, (2.2)). By Chapter II, (7.1) and (7.4), the orthogonal complement to the orbit of A, at A, is precisely $\mathfrak{p}_A$, which is equal to $\mathfrak{t}$. Hence all the elements of $S(\mathfrak{p};A,N)$ lie in $\mathfrak{t}$. Conversely, every line in $\mathfrak{t}$ starting at A is transversal. This proves the proposition.

COROLLARY 2.5. *Every orbit* N *of* K *on* $\mathfrak{p}$ *intersects* $\mathfrak{t}$.

Because every such orbit is compact, there must be a point at which N is closest to A. The line from A to that point will then have to be transversal. Hence $S(\mathfrak{p};A,N)$ is not empty. Now apply Proposition 2.4.

COROLLARY 2.6. *Any two Cartan algebras of* (G,K) *are conjugate under the adjoint action of* K.

Proof. Let $\mathfrak{t}'$ be any Cartan algebra of (G,K), let B be a general point of $\mathfrak{t}'$ and let N be the orbit of B. Since $N\cap\mathfrak{t}\neq 0$ there exists a $k\in K$, such that $\mathrm{Ad}k\cdot B\subset\mathfrak{t}$. But then clearly $\mathrm{Ad}k\cdot\mathfrak{t}'=\mathfrak{t}$, so that $\mathfrak{t}'$ is conjugate to $\mathfrak{t}$.

Note. This corollary goes back to E. Cartan [9]. The proof given here is an immediate extension of the proof due to Hunt for the conjugacy of Cartan algebras in the group case [13].

PROPOSITION 2.7. (a) *Every orbit of* K *in* $\mathfrak{p}$ *intersects* $\mathfrak{t}$ *orthogonally.* (b) *The orbit through* A *has maximal dimension.*

Proof. Clearly $\mathfrak{p}_B\supset\mathfrak{p}_A=\mathfrak{t}$ for any $B\in\mathfrak{t}$. As the transversal space to the orbit through B is $\mathfrak{p}_B$, this inclusion together with Corollary 2.5 proves both statements.

We write δ_K' for the defect function of the adjoint action of K on $\mathfrak{p}$, as defined in Chapter I, (7.4). Because of (2.7(b)) and Chapter II, (7.1), we have

$$\delta_K'(B)=\dim[\mathfrak{k},A]-\dim[\mathfrak{k},B]. \tag{2.8}$$

Applying (2.1(b)), (2.2), we obtain

$$\delta_K'(B) = \dim \mathfrak{p}_B - \dim \mathfrak{p}_A = \dim \mathfrak{k}_B - \dim \mathfrak{k}_A.$$

This proves

PROPOSITION 2.9.

$$\delta_K'(B) = \tfrac{1}{2}(\dim \mathfrak{g}_B - \dim \mathfrak{g}_A)$$

for any $B \in \mathfrak{p}$.

According to Chapter I, (7.3), the defect δ_s of any segment

$$s \in S(\mathfrak{t}; A, N \cap \mathfrak{t}),$$

with end points A and Q, is given by

$$\delta_s = \sum_{0 \leqq \alpha < 1} \delta_K'((1-\alpha)A + \alpha Q). \tag{2.10}$$

We extend $\mathfrak{t}$ to a Cartan algebra $\tilde{\mathfrak{t}}$ of the Lie algebra $\mathfrak{g}$, and denote by $D'(G)$ the infinitesimal diagram of G on $\mathfrak{t}$ (cf. Chapter III, no. 1).

As a consequence of (2.9), δ_K' can be described entirely in terms of $D'(G)$. In fact, it is easily seen that

$$\delta_K'(B) = \textit{number of planes of } D'(G) \textit{ containing } B, \textit{ but not containing all of } \mathfrak{t}. \tag{2.11}$$

(For each root $\theta \in \mathcal{R}$ with $\theta(B) = 0$, the dimension of $\mathfrak{g}_B$ goes up by two.)

The set of points B in $\mathfrak{t}$ with $\delta_K'(B) > 0$ is called the infinitesimal diagram of (G, K), and is denoted by $D'(G, K)$; it consists of a finite number of planes through the origin. The defect δ_s of a segment s is the number of planes, properly weighted, in $D'(G, K)$ crossed by s; note that planes of $D'(G, K)$ containing the end point Q of s do not count in δ_s.

The connected components of the complement of $D'(G, K)$ in $\mathfrak{t}$ are called the *fundamental chambers* of $D'(G, K)$. In complete analogy to the group case, every chamber $\mathcal{F}$ plays the role of a fundamental domain for the Weyl group of (G, K), as will be seen later. This group, denoted by $\mathcal{W}(G, K)$, is defined as the quotient of the group consisting of the elements in K which keep $\mathfrak{t}$ setwise fixed (the normalizer of $\mathfrak{t}$) modulo the group of the elements which keep $\mathfrak{t}$ pointwise fixed (i.e. $K_{\mathfrak{t}}$). From Proposition 2.7(a), applied to A (and the fact that all orbits are compact), it follows that $\mathcal{W}(G, K)$ is a finite group. It is also clear that if B is any general point of $\mathfrak{t}$ and N is the orbit of B, then

$$N \cap \mathfrak{t} = \textit{orbit of } B \textit{ under } \mathcal{W}(G, K). \tag{2.12}$$

(Actually, a more general statement is true: if A, B are two subsets of $\mathfrak{t}$, conjugate under $k \in K$, then there exists an element of the Weyl group, whose restriction to A equals that of $\mathrm{Ad}k$. One proves this by considering the centralizer, in G, of $\mathfrak{t} \cap \mathrm{Ad}k(\mathfrak{t})$. A similar statement holds for the global case, i.e., for the operation of the Weyl group on the maximal torus T, cf. Definition 3.3.)

For any K-orbit N on $\mathfrak{p}$, we write $\bar{S}_*(N)$ for the graded $\mathbf{Z}_2$-vector space generated by the elements of $S(\mathfrak{t}; A, \mathfrak{t} \cap N)$, the dimension of a segment being the defect δ_s (cf. (2.10)).

We can now state the consequences of Theorems I and II in the following form:

THEOREM VI. *Let N be any orbit of K on $\mathfrak{p}$. Then (as graded modules)*

$$H_*(N; \mathbf{Z}_2) \approx \bar{S}_*(N).$$

Proof. We have $\Omega(\mathfrak{p}; A, N) \approx N$ because $\mathfrak{p}$ is a Euclidean space (cf. Chapter III, no. 6). By Theorem II, the action of K on $\mathfrak{p}$ is variationally complete, and the point A has defect 0. Hence Theorem I is applicable, and describes $H_*(N; \mathbf{Z}_2)$ in terms of $S(\mathfrak{p}; A, N)$, or, because of (2.4), in terms of $S(\mathfrak{t}; A, \mathfrak{t} \cap N)$, graded by δ_s.

Remarks. This theorem is the complete analogue of Theorem III in [6], except that the coefficients have been reduced mod 2. One can of course give the data for $\bar{S}_*(N)$ entirely in terms of the roots of G, and the group $\mathfrak{W}(G, K)$. Such a formulation was given in [6]. We omit these details here, and give only one example in the next corollary.

COROLLARY 2.13. *Let $\mathfrak{W}_*$ be the graded $\mathbf{Z}_2$-vector space generated by the elements of $\mathfrak{W}(G, K)$, the dimension of $w \in \mathfrak{W}(G, K)$ being*

$$\sum_{0<\alpha\leq 1} \delta_K'\{(1-\alpha)A + \alpha(w \cdot A)\}.$$

Let $K_{\mathfrak{t}}$ be the centralizer of $\mathfrak{t}$ in K. Then as graded modules,

$$H_*(K/K_{\mathfrak{t}}; \mathbf{Z}_2) = \mathfrak{W}_*$$

Proof. The elements of $\mathfrak{W}(G, K)$ clearly permute the fundamental chambers of $D'(G, K)$, since the function δ_K' is invariant under $\mathfrak{W}(G, K)$. We need the following proposition:

PROPOSITION 2.14. *The Weyl group $\mathfrak{W}(G, K)$ is transitively and faithfully represented by the permutations it induces in the fundamental chambers of $D'(G, K)$.*

We prove this proposition by means of Theorem VI. Let $\mathfrak{F}$ be any fundamental chamber of $D'(G,K)$, and let B be a general point in it with K-orbit N. Let $\mathfrak{F}_A$ be the chamber that contains A. By (2.10), (2.11), (2.12), the number of points in the $\mathfrak{W}(G,K)$-orbit of B that lie in $\mathfrak{F}_A$ equals the number of segments in $S(\mathfrak{t};A,\mathfrak{t}\cap N)$ of index 0. By Theorem VI, this is also the number of components of N. But K being connected, N is connected, and (2.14) is proved.

The Corollary 2.13 is now immediate. Let B be taken in $\mathfrak{F}_A$. With any $w\in\mathfrak{W}(G,K)$, associate the segment from A to $w\cdot B$. This induces a gradation preserving isomorphism of $S_*(N)$ and $\mathfrak{W}_*$. Since B is general in $\mathfrak{t}$, its K-orbit N is homeomorphic to $K/K_{\mathfrak{t}}$. The corollary follows now from Theorem VI.

Let $\mathfrak{t}'$ be any subspace of $\mathfrak{t}$. A result analogous to (2.13) can then be stated for the homology of $K/K_{\mathfrak{t}'}$ in terms of the quotient set of $\mathfrak{W}(G,K)$ by the subgroup that leaves $\mathfrak{t}'$ point-wise fixed.

3. The action of $K\times K$ on G. This section is the global analogue of the previous one. As in Chapter II, we write $\mathfrak{k}_a=\mathfrak{k}\cap \mathrm{Ad}a^{-1}\mathfrak{k}$, $\mathfrak{p}_a=\mathfrak{p}\cap \mathrm{Ad}a^{-1}\mathfrak{p}$ for any $a\in G$. We also write $\mathfrak{g}_a$ for $\{X\in g\colon \mathrm{Ad}aX=X\}$; note that the definition of $\mathfrak{g}_a$ differs from that of $\mathfrak{k}_a$ and $\mathfrak{p}_a$.

PROPOSITION 3.1. *If $a^*=a^{-1}$, then*

$$\mathfrak{g}_{a^2}=\mathfrak{k}_a\oplus\mathfrak{p}_a \tag{a}$$

$$\mathfrak{p}=\{\mathrm{Ad}a-\mathrm{Ad}a^{-1}\}\,\mathfrak{k}\oplus\mathfrak{p}_a \tag{b}$$

are orthogonal decompositions.

Proof. Clearly, $\mathfrak{g}_{a^2}$ is stable under *. Hence $\mathfrak{g}_{a^2}=\mathfrak{g}_{a^2}\cap\mathfrak{k}+\mathfrak{g}_{a^2}\cap\mathfrak{p}$. If $Z\in\mathfrak{p}_a$, then $\mathrm{Ad}aZ\in\mathfrak{p}$. Hence, applying *, $\mathrm{Ad}a^{-1}Z=\mathrm{Ad}aZ$, whence $Z\in\mathfrak{g}_{a^2}\cap\mathfrak{p}$. Hence $\mathfrak{p}_a\subset\mathfrak{g}_{a^2}\cap\mathfrak{p}$. Conversely, if $Z\in\mathfrak{g}_{a^2}\cap\mathfrak{p}$, then $(\mathrm{Ad}a^{-1}Z)^*=-\mathrm{Ad}aZ=-\mathrm{Ad}a^{-1}Z$, so that $Z\in\mathfrak{p}_a$. Hence $\mathfrak{p}\supset\mathfrak{g}_{a^2}\cap\mathfrak{p}$. Quite similarly, one finds that $\mathfrak{g}_{a^2}\cap\mathfrak{k}=\mathfrak{k}_a$. This proves 3.1(a). Next, $\{(\mathrm{Ad}a-\mathrm{Ad}a^{-1})Z\}^*=-(\mathrm{Ad}a-\mathrm{Ad}a^{-1})Z^*$. Hence, $(\mathrm{Ad}a-\mathrm{Ad}a^{-1})\mathfrak{k}\subset\mathfrak{p}$, and $(\mathrm{Ad}a-\mathrm{Ad}a^{-1})\mathfrak{p}\subset\mathfrak{k}$. Suppose that $Z\in\mathfrak{p}$ is perpendicular to $(\mathrm{Ad}a-\mathrm{Ad}a^{-1})\mathfrak{k}$, so that

$$(\{\mathrm{Ad}a-\mathrm{Ad}a^{-1}\}\mathfrak{k},Z)=0.$$

This implies $(\mathrm{Ad}a^{-1}-\mathrm{Ad}a)Z\in\mathfrak{p}$, whence $\mathrm{Ad}a^{-1}Z=\mathrm{Ad}aZ$, so that $Z\in\mathfrak{g}_{a^2}\cap\mathfrak{p}=\mathfrak{p}_a$. The argument can be reversed, and (3.1(b)) is proved.

COROLLARY 3.2. *If $a^*=a^{-1}$,* then

$$\dim \mathfrak{g}_{a^2} \cap \mathfrak{p} = \dim \mathfrak{g}_{a^2} \cap \mathfrak{k};$$

$$\dim \mathfrak{k}_a - \dim \mathfrak{p}_a = \dim \mathfrak{k} - \dim \mathfrak{p}.$$

This follows from (3.1(a), (b)) as (2.2) followed from (2.1(a), (b)) in no. 2; first one has to verify, using *, that $\mathfrak{k}_a$ is the kernel of the restriction of $\mathrm{Ad}a - \mathrm{Ad}a^{-1}$ to $\mathfrak{k}$.

DEFINITION 3.3. *The image in G, under the exponential map, of a Cartan algebra $\mathfrak{t}$ of (G,K) is called a maximal torus of (G,K), and is denoted by T. The point $a \in T$ is called general if $\mathfrak{p}_a = \mathfrak{t}$.*

The maximality of $\mathfrak{t}$ implies that $\exp \mathfrak{t}$ is closed in G. By general theory, we can find an $a \in T$ such that $\mathfrak{g}_{a^2} \cap \mathfrak{p} = \mathfrak{t}$. Hence by (3.1), T contains general points.

In the sequel, T denotes a fixed maximal torus of (G,K) and a is a fixed general point of T. The tangent space to T at the identity is identified with the Cartan algebra $\mathfrak{t}$.

Let $b \in T$, and let $N = K \cdot b \cdot K$ be the orbit of b under $K \times K$. By Chapter II, (7.1), (7.4) the transversal space to N at b is precisely $b \cdot \mathfrak{p}_b$. The arguments of no. 2 therefore easily yield the following analogue of (2.4), (2.5), and (2.7).

PROPOSITION 3.4. *Let N be any orbit of $K \times K$ on G. Then $N \cap T$ is not vacuous. N intersects T orthogonally (and therefore at a finite number of places). The sets*

$$S(G;a,N) \text{ and } S(T;a,T \cap N)$$

coincide. The orbit of a has maximal dimension.

Here $S(G;a,N)$ of course denotes the set of transversal geodesics from a to N, while $S(T;a,T \cap N)$ stands for the, automatically transversal, geodesics on T from a to points of $N \cap T$.

We denote by δ_K the defect function of $K \times K$ on G, defined as in Chapter I, (7.1). By Chapter II, no. 3 we have $\delta_K(b) = \dim \mathfrak{k}^a - \dim \mathfrak{k}^b$. As in no. 2, one proves that

$$\delta_K(b) = \tfrac{1}{2}(\dim \mathfrak{g}_{b^2} - \dim \mathfrak{g}_{a^2}) \text{ for any } b \in T. \tag{3.5}$$

(Note that for such b's the relation $b^* = b^{-1}$ holds.)

As before, $\tilde{\mathfrak{t}}$ is a Cartan algebra of g, containing $\mathfrak{t}$; let $D(G)$ be the diagram of G on $\tilde{\mathfrak{t}}$ (cf. Chapter III, no. 1). We define the *defect function* δ of (G,K) by

(3.6) $\delta(B) =$ *number of planes of* $D(G)$ *that contain* B, *but do not contain all of* $\mathfrak{t}$, *for any* $B \in \mathfrak{t}$.

The diagram $D(G,K)$ consists now by definition of all $B \in \mathfrak{t}$ with $\delta(B) > 0$; it can also be described as the union of the intersections $\mathfrak{p} \cap \mathfrak{t}$ for all those singular planes $p = (\theta, n) \in \mathfrak{P}$, whose translation vector τ_θ is not orthogonal to $\mathfrak{t}$.

The defects δ and δ_K are related by the following proposition.

PROPOSITION 3.7. *Let* $\rho: \mathfrak{t} \to T$ *be defined by* $\rho(B) = \exp(\frac{1}{2}B)$. *Then* $\delta_K \circ \rho = \delta$, *where* δ_K *is the defect function of* $K \times K$ *on* G, *and* δ *is the defect function of* $D(G,K)$.

This formula is an immediate consequence of (3.5), and the meaning of $D(G)$ for the stabilizers of points in T. Let A be an element of $\mathfrak{t}$ that maps into a under ρ; clearly A is general in $\mathfrak{t}$. For any $K \times K$-orbit N, the elements of $S(T; a, T \cap N)$ then lift under ρ uniquely into straight line segments in $\mathfrak{t}$, with initial point A. Let $S'(N) = S'(\mathfrak{t}; A, \rho^{-1}(T \cap N))$ denote this set of line segments. Let s', with end points A and B, be an element of $S'(N)$, corresponding to $s \in S(T; a, T \cap N)$. Then the δ_K-defect of s is equal to the δ-defect of s', given by

$$\delta_{s'} = \sum_{0 \leqq \alpha < 1} \delta((1-\alpha)A + \alpha B), \tag{3.8}$$

the number of planes, properly weighted, of $D(G,K)$ crossed by s'. Finally, let $\bar{S}'_*(N)$ be the $\mathbf{Z}_2$-vector space generated by $S'(N)$ according to the defect δ. The global analogue of Theorem VI now clearly takes the following form:

THEOREM VII. *Let* N *be any orbit of* $K \times K$ *on* G. *Then, as graded modules,*

$$H_*\{\Omega(G; a, N); \mathbf{Z}_2\} \approx \bar{S}'_*(N).$$

The Corollary 2.13 to Theorem VI also has a global analogue. We will formulate it in terms of the cells of $D(G,K)$. By definition, these are the connected components of $\mathfrak{t} - D(G,K)$. We let $\mathfrak{F}$ be a fundamental chamber of $D'(G,K)$ and denote by $\{\Delta\}_{\mathfrak{F}}$ the set of cells of $D(G,K)$ contained in the closure of $\mathfrak{F}$. We assign an index $\delta(\Delta)$ to $\Delta \in \{\Delta\}_{\mathfrak{F}}$ by the formula

$$\delta(\Delta) = \sum_{0 < t \leqq 1} \delta(tB), \tag{3.9}$$

where δ is the defect function of (G,K) and B is any point of Δ. (In words, $\delta(\Delta) =$ number of properly counted planes in $D(G,K)$ crossed by a line

from 0 to a point of Δ.) Finally, we let $\mathfrak{F}_*$ be the graded $\mathbf{Z}_2$-vector space generated by $\{\Delta\}_\mathfrak{F}$ and graded by δ.

COROLLARY 3.10. *Let* $\pi_1(G) = 0$. *Then,*

$$H_*(\Omega(G/K); \mathbf{Z}_2) \approx \mathfrak{F}_*$$

as graded modules.

Proof. Because $\pi_1(G) = 0$ and $\pi_0(K) = 0$, G/K is simply connected. Hence $\Omega(G/K)$ has precisely one component. Let $N = K$ be the orbit of $e \in G$ under $K \times K$. Then, as explained in no. 1, $\Omega(G/K)$ can be identified with $\Omega(G; a, K)$. Let $\mathfrak{F}_0 = \rho^{-1}(K \cap T)$, so that $S'(K)$ consist of the segments from A to the points of $\mathfrak{F}_0$. Then Theorem VII implies that $\mathfrak{F}_0$ intersects the closure of any cell Δ in $D(G, K)$ precisely once. To show this, we may assume $A \in \Delta$. Since $\Omega(G; a, K)$ is connected, by Theorem VII there is only one segment in $S'(K)$ of index 0; but this is equivalent to the assertion about $\mathfrak{F}_0$.

Now to any $s \in S'(K)$, with second end point $B \in \mathfrak{F}_0$, there exists a unique $w \in \mathfrak{W}(G, K)$ such that $w \cdot (A - B)$ lies in the interior of a cell in $\mathfrak{F}$. As in [6], one shows that this defines a one-to-one map of $S'(K)$ onto $\{\Delta\}_\mathfrak{F}$, and that the induced isomorphism of $\bar{S}'_*(K)$ onto $\mathfrak{F}_*$ is gradation preserving. The Corollary now follows from Theorem VII.

As a simple application, we will prove the following proposition, which, as the referee pointed out, is contained in [9], no. 101.

PROPOSITION 3.11. *Let* * *be an involution of the compact, simply connected Lie group* G. *Then the set of fixed points of* * *is connected.*

Proof. Let (G, K) be the symmetric pair determined by *, and let $\hat{K}$ be the fixed point set of *. Thus K is the e-component of $\hat{K}$. We start with the following lemma.

LEMMA 3.12. *If* $K \cap T = \hat{K} \cap T$, *then* $K = \hat{K}$.

Proof. Suppose K_1 is a component of $\hat{K}$ different from K. The $K \times K$-orbit of any point e_1 of K_1 is clearly K_1 itself. By (3.4), this orbit intersects T, so that $K_1 \cap T$ is not vacuous. But $K \cap T$ and $K_1 \cap T$ are disjoint, and (3.12) follows. Next, we remark:

(3.13) *The set* $\hat{K} \cap T$ *consists of the points* $x \in T$, *with* $x^2 = e$.

This follows from the fact that on T the involution * is just inversion.

LEMMA 3.14. *If* $\pi_1(G) = 0$, *then* $K \cap T = \hat{K} \cap T$.

Proof. Let $\mathcal{J}_0 = \rho^{-1}(K \cap T)$ as before, and let $\mathcal{J} = \rho^{-1}(\hat{K} \cap T)$. It follows from (3.13) and the definition $\rho(B) = \exp(\frac{1}{2}B)$ that $\mathcal{J}$ consists of the lattice of points in $\mathfrak{t}$ which under the exponential map go into e. We shall show that $\mathcal{J}_0 = \mathcal{J}$. Suppose then that $\mathcal{J}$ is a proper refinement of $\mathcal{J}_0$. Since the closed cells of $D(G,K)$ cover $\mathfrak{t}$, and since $\mathcal{J}_0$ intersects each closed cell of $D(G,K)$ in precisely one point (as noted in the proof of 3.10), the lattice $\mathcal{J}$ will have to intersect the closure of some cell, say Δ, in at least two points. From the construction of $D(G,K)$, it is clear that Δ is contained in the closure of some cell Δ_1 of $D(G)$ in $\bar{\mathfrak{t}}$. (Here $\bar{\mathfrak{t}}$ is a Cartan algebra of G containing $\mathfrak{t}$.) Hence the closure of Δ_1 in $D(G)$ intersects $\mathcal{J}$ in at least two points. But it is well known that if $\pi_1(G) = 0$, then each closed cell of $D(G)$ contains exactly one point whose exponential image is e (cf. [20]). This contradiction proves (3.14). The two Lemmas 3.12, 3.14 prove Proposition 3.11.

4. An example. Symmetric spaces of maximal rank. According to (3.10), the mod 2 Poincaré series of $\Omega(G/K)$ is given by $\sum_{\mathcal{F}} t^{\delta(\Delta)}$, where Δ runs over the cells of a fundamental chamber $\mathcal{F}$ in $D(G,K)$. In Theorem B, Part 1, of [6], the Poincaré series of $\Omega(G)$ was given as $\sum_{\mathcal{F}} t^{2\lambda(\Delta)}$, where one now sums over the cells of a fundamental chamber in $D(G)$. Suppose now that the maximal torus of (G,K) is also a maximal torus of G. In this case, $D(G)$ coincides with $D(G,K)$, and $\delta(\Delta)$ as defined here, agrees with the $\lambda(\Delta)$ of [6]. We therefore get the following somewhat mysterious application of 3.10.

PROPOSITION 4.1. *Suppose that $\pi_1(G) = 0$, and that the rank of the symmetric pair (G,K) equals the rank of G. Then*

$$\dim H_q(\Omega(G/K); \mathbf{Z}_2) = \dim H_{2q}(\Omega(G); \mathbf{Z}_2).$$

Every 1-connected Lie group has an essentially unique involution $*$, with rank $(G,K) =$ rank G. This involution correspond to "the" real form of the complexified Lie group. We will call a G/K obtained in this way a symmetric space of maximal rank. The mod 2 Betti numbers of $\Omega(G/K)$ are then determined by the Betti numbers of $\Omega(G)$ by (4.1); on the other hand, the Poincaré series of $\Omega(G)$ is determined by the Poincaré polynomial of G (cf. [6]). Indeed, if

$$P(G;t) = (1 + t^{2m_1-1}) \cdots (1 + t^{2m_l-1}),$$

then

$$P(\Omega(G);t) = (1 - t^{2(m_1-1)})^{-1} \cdots (1 - t^{2(m_l-1)})^{-1}.$$

Combining this observation with (4.1), we obtain the following proposition.

PROPOSITION 4.2. *Let G/K be a symmetric space of maximal rank, where G is a 1-connected compact Lie group. If $m_1, \cdots, m_l$ are the primitive exponents of G, then*

$$P(\Omega(G/K); \mathbf{Z}_2; t) = (1-t^{m_1-1})^{-1}(1-t^{m_2-1})^{-1} \cdots (1-t^{m_l-1})^{-1}$$

A similar comparison of (2.15) with Theorem B, Part 2 of [6], leads to the following proposition.

PROPOSITION 4.3. *Let (G,K) be a symmetric pair of maximal rank. Let T_1 be a torus of G contained in a maximal torus of (G,K). Let K_{T_1} and G_{T_1} denote the centralizers of T_1 in K and G respectively. Then*

$$\dim H_q(K/K_{T_1}; \mathbf{Z}_2) = \dim H_{2q}(G/G_{T_1}; \mathbf{Z}_2).$$

As a concrete instance of the halving in dimensions, we mention the real projective space as opposed to the complex projective space.

Applying the Hirsch formula [1] to the right hand side of (4.3), with $T_1 = T$, one obtains the following analogue of (4.2).

PROPOSITION 4.4. *Let (G,K) be a symmetric pair of maximal rank; let T be a maximal torus of (G,K); let $m_1, \cdots, m_l$ be the primitive exponents of G. Then*

$$P(K/K_T; \mathbf{Z}_2; t) = (1-t^{m_1}) \cdots (1-t^{m_l})/(1-t)^l.$$

In certain cases, for instance when $G = SU(n)$ and $K = SO(n)$, the formulas (4.3) and (4.4) also follow from Borel's work with the maximal tori mod 2 of G [2].

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INDEX

PAGE

ANDERSON, R. D. The algebraic simplicity of certain groups of homeomorphisms, ... 955

ANDREOTTI, ALDO. On a theorem of Torelli, ... 801

AURORA, SILVIO. On power multiplicative norms, ... 879

BAILY, WALTER L., JR. Satake's compactification of V_n, . . . 348

BARCUS, W. D. and J.-P. MEYER. The suspension of a loop space, . 895

BRAUER, FRED and SHLOMO STERNBERG. Local uniqueness, existence in the large, and the convergence of successive approximations, 421

BERBERIAN, S. K. $N \times N$ matrices over an AW^*-algebra, . . 37

BOREL, A. and F. HIRZEBRUCH. Characteristic classes and homogeneous spaces, I, ... 458

BOTT, RAOUL and HANS SAMELSON. Applications of the theory of Morse to symmetric spaces, ... 964

BROWDER, FELIX E. Eigenfunction expansions for non-symmetric partial differential operators, I, ... 365

CALDERÓN, A. P. Uniqueness in the Cauchy problem for partial differential equations, ... 16

CARLESON, LENNART. An interpolation problem for bounded analytic functions, ... 921

CARTIER, P. et J. DIXMIER. Vecteurs analytiques dans les représentations de groupes de Lie, ... 131

CHOW, WEI-LIANG. Remarks on my paper "The Jacobian variety of an algebraic curve, ... 238

CHOW, WEI-LIANG. The criterion for unit multiplicity and a generalization of Hensel's lemma, ... 539

CORSON, H. H. The determination of paracompactness by uniformities, ... 185

DEVINATZ, A. and I. I. HIRSCHMAN, JR. The spectra of multiplier transforms on l^p, ... 829

DIEUDONNÉ, JEAN. Lie groups and Lie hyperalgebras over a field of characteristic $p > 0$, ... 740

DIXMIER, J. See P. Cartier et J. Dixmier, page 131.

PAGE

GERSTENHABER, MURRAY. On nilalgebras and linear varieties of nilpotent matrices, I, ... 614

HARISH-CHANDRA. Spherical functions on a semisimple Lie group, I, 241

HARISH-CHANDRA. Spherical functions on a semisimple Lie group, II, 553

HIRSCHMAN, I. I., JR. See A. Devinatz and I. I. Hirschman, Jr., page 829.

HIRZEBRUCH, F. See A. Borel and F. Hirzebruch, page 458.

IN MEMORIAM. Aurel Wintner, 1903-1958, ... 1

IWASAWA, KENKICHI. On some invariants of cyclotomic fields, . 773

KERVAIRE, MICHEL. On the Pontryagin classes of certain $SO(n)$, bundles over manifolds, ... 632

KOLCHIN, ELLIS and SERGE LANG. Algebraic groups and the Galois theory of differential fields, ... 103

LANG, SERGE. See Ellis Kolchin and Serge Lang, page 103.

LANG, SERGE. Reciprocity and correspondences, ... 431

LANG, SERGE and JOHN TATE. Principal homogeneous spaces over Abelian varieties, ... 659

MATSUSAKA, T. Polarized varieties, fields of moduli and generalized Kummer varieties of polarized Abelian varieties, . . 45

MATSUSAKA, T. On a theorem of Torelli, ... 784

MAUTNER, F. I. Spherical functions over $\mathfrak{P}$-adic fields, I, . . 441

MEYER, J.-P. See W. D. Barcus and J.-P. Meyer, page 895.

MORREY, CHARLES B., JR. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. Part I, ... 198

MORREY, CHARLES B., JR. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. Part II, ... 219

MURRE, J. P. On a connectedness theorem for a birational transformation at a simple point, ... 3

MURRE, J. P. Intersection multiplicites of maximal connected bunches, 311

MOSTOW, G. D. Extensions of representations of Lie groups, II, . 331

NAGATA, MASAYOSHI. A general theory of algebraic geometry over Dedekind domains, II, ... 382

NASH, J. Continuity of solutions of parabolic and elliptic equations, 931

PAGE

O'MEARA, O. T. The integral representations of quadratic forms over local fields, ... 843

ROSENLICHT, MAXWELL. Extensions of vector groups by Abelian varieties, ... 685

SAMELSON, HANS. See Raoul Bott and Hans Samelson, page 964.

SCHLESINGER, ERNEST C. Conformal invariants and prime ends, . 83

SERRE, JEAN-PIERRE. Quelques propriétés des variétés abéliennes en en caractéristique p, ... 715

STERNBERG, SHLOMO. See Fred Brauer and Shlomo Sternberg, page 421.

STERNBERG, SHLOMO. On the structure of local homeomorphisms of Euclidean n-space, II, ... 623

TAMAGAWA, TSUNEO. On the structure of orthogonal groups, . . 191

TATE, JOHN. See Serge Lang and John Tate, page 659.

WINTNER, AUREL. Stratifications of Cauchy's "stable" transcendents of Mittag-Leffler's entire functions, ... 111

WINTNER, AUREL. On cosine-like arches, ... 125

WINTNER, AUREL. In the λ-variant of Mertens' μ-hypothesis, . . 639

WRIGHT, E. M. Partitions of large bipartites, ... 643

ZARISKI, OSCAR. The problem of minimal models in the theory of algebraic surfaces, ... 146

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INDEX

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